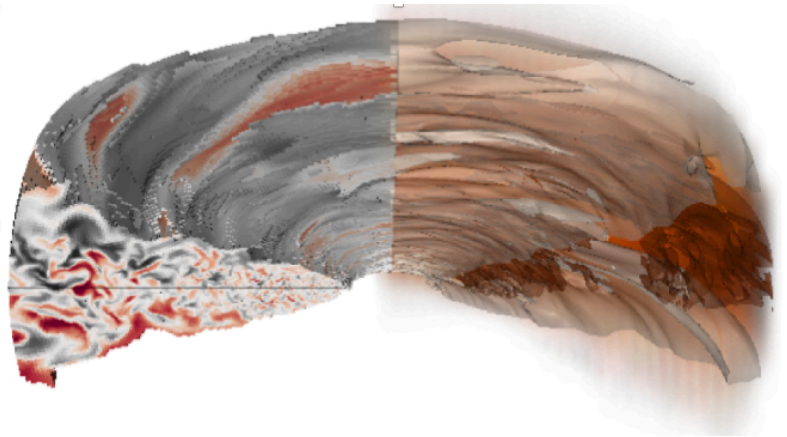


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Astrophysical Fluid Dynamics: II. Magnetohydrodynamics

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Outline

- Astrophysical fluids as plasmas
- The MHD formulation
- Conservation laws and physical interpretation
- Generalized Ohm's law, and limitations of MHD
- MHD waves
- MHD shocks and discontinuities
- MHD instabilities (examples)

Outline

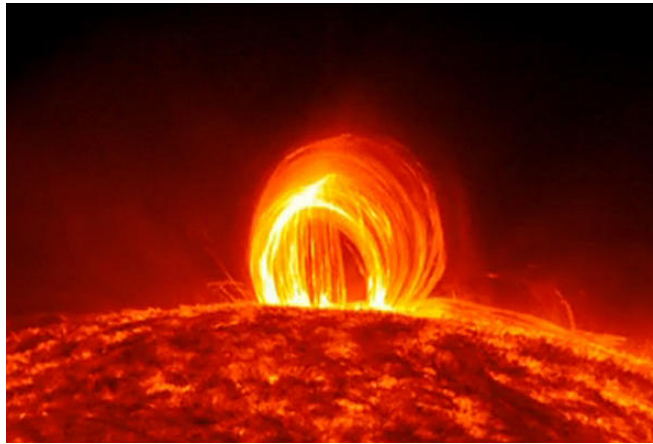
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What is a plasma?

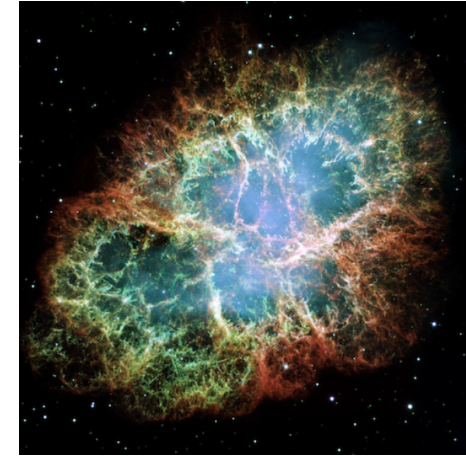
Plasma is a state of matter comprising of fully/partially ionized gas.



Lightening



The restless Sun



Crab nebula

A plasma is generally quasi-neutral and exhibits collective behavior.

Net charge density averages to zero on relevant scales (i.e., Debye length).

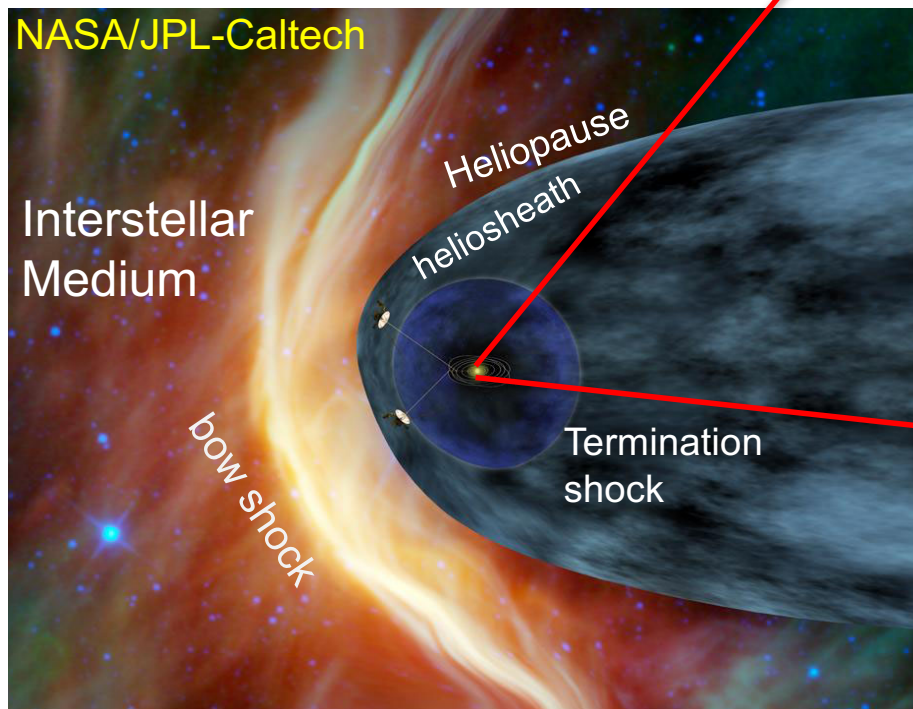
particles interact with each other at long-range through electromagnetic fields (plasma waves).

Why plasma astrophysics?

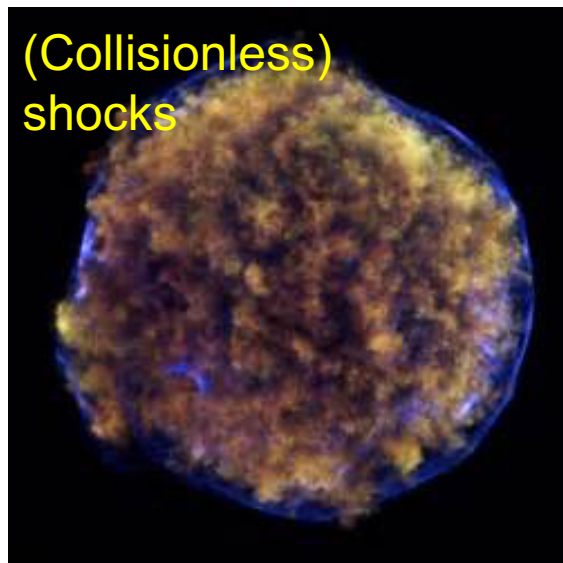
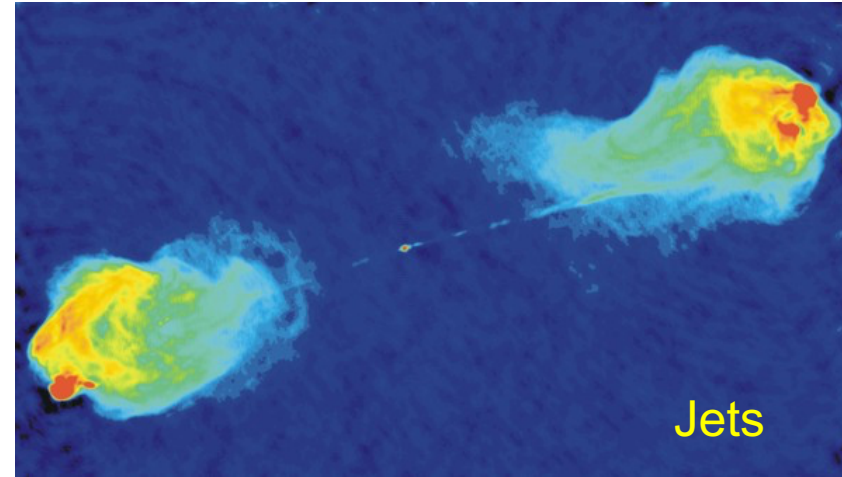
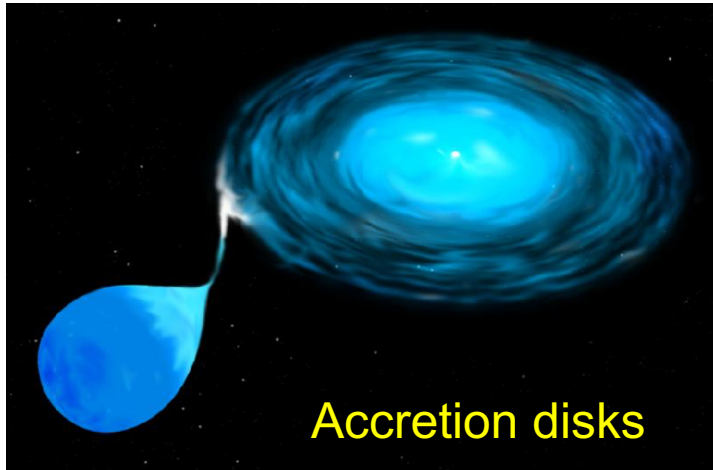
- More than 99.9% of observable matter in the universe is plasma.
- Magnetic fields play vital roles in many astrophysical processes.
- Plasma astrophysics allows the study of plasma phenomena at extreme regions of parameter space that are in general inaccessible in the laboratory.

Heliophysics and space weather

- Solar physics (including flares, coronal mass ejection)
- Interaction between the solar wind and Earth's magnetosphere
- Heliospheric physics



Astrophysical applications



Comparison of plasma and gas phases

Property	Gas	Plasma
Electrical conductivity	Very low	Usually very high (effectively infinite in most cases)
Independently acting species	Usually, one	Two or three (electrons, ions and sometimes neutrals)
Velocity distribution	Maxwellian (due to frequent collisions)	Often non-Maxwellian: many plasmas are collisionless
Interactions	Binary collisions	Collective: organized motion by interacting with long-range electromagnetic fields in the form of plasma waves.

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How to describe a plasma?

- The exact microscopic description
 - Follow the trajectories of all particles, and solve for the evolution of EM fields.
 - The description is exact, but is computationally hopeless.
 - Solutions can be chaotic (depending on initial conditions).
- Kinetic theory (still microscopic)
 - Use statistical approach to describe particles (i.e., distribution function).
 - Evolution of the distribution function according to particle orbit theory.
 - EM fields are largely long-range, whereas weak binary interactions can be treated as “collisions”.
- Fluid theory (macroscopic)
 - Only focus on macroscopic fluid quantities (e.g., density, bulk velocity, T).
 - No need to follow individual particle orbits.

Magneto-hydrodynamics (MHD)

MHD couples **Maxwell's equations** with **hydrodynamics** to describe the **macroscopic** behavior of highly conducting fluid such as plasmas.

Ideal MHD involves several important approximations, as we list below.

1. Flow velocity is very non-relativistic.

Can be relaxed to formulate *relativistic MHD*.

2. Electric conductivity is so high that can be considered as infinite.

Can be relaxed to formulate *non-ideal MHD*.

3. Low-frequency, long-wavelength.

This is the key to the fluid description of plasmas (besides collisionalities).

We first formulate MHD equations from the above approximations, and then justify them (for 2 and 3).

Reduction of Maxwell's equations

We focus on the non-relativistic regime where flow velocity $v \ll c$.

Starting from Maxwell's equations:

[1] ~~$\nabla \cdot \mathbf{E} = 4\pi\rho_e$~~ , (implied from [4])

[2] $\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}$, $\Rightarrow \frac{E}{B} \sim \frac{L}{cT} \sim \frac{V}{c}$

[3] ~~$\nabla \cdot \mathbf{B} = 0$~~ , (implied from [2])

[4] $\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}$. Displacement current can be dropped.

$\sim B/L$

$\sim E/cT \sim (V^2/c^2) (B/L)$

$\Rightarrow \mathbf{J} = \frac{c}{4\pi} \nabla \times \mathbf{B}$

The system can adjust its current adjusted instantaneously to match field configuration.

The induction equation

B field evolves according to the **induction equation**:

$$\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \quad \text{But what determines } \mathbf{E}?$$

Under the assumption that the fluid is **infinitely conducting**:

$$\mathbf{E}' = 0 \quad \text{(in fluid rest frame)}$$

This leads to **ideal MHD**.

The relation between \mathbf{E} and \mathbf{E}' is given by a Lorentz transformation:

$$\mathbf{E}' = \frac{\mathbf{E} + (\mathbf{v}/c) \times \mathbf{B}}{\sqrt{1 - v^2/c^2}} \approx \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}$$

where again we have assumed $v \ll c$.

With $\mathbf{E}'=0$, the induction equation becomes

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

Momentum equation

A conducting fluid is further subject to the Lorentz force:

$$\mathbf{F}_{\text{EM}} = \frac{1}{c} \mathbf{J} \times \mathbf{B} + \rho_e \mathbf{E}$$

$\sim B^2/L$ $\sim E^2/L \sim (V^2/c^2)B^2/L$

However, the electric force about a factor $(V/c)^2$ smaller and can be dropped.

Therefore, the MHD momentum equation simply read:

$$\begin{aligned} \rho \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} &= -\nabla P - \nabla \Phi + \frac{1}{c} \mathbf{J} \times \mathbf{B} \\ &= -\nabla P - \nabla \Phi + \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B} \end{aligned}$$

Summary: ideal MHD equations

Continuity equation (unchanged):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Momentum equation (now includes the Lorentz force):

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = -\nabla P - \nabla \Phi + \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B}$$

Induction equation (new addition):

$$\frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

Thermal energy equation (unchanged in the current form):

$$\frac{Ds}{Dt} = \frac{\partial s}{\partial t} + (\mathbf{v} \cdot \nabla) s = 0 \quad \text{or} \quad \frac{\partial P}{\partial t} + \mathbf{v} \cdot \nabla P + \gamma P \nabla \cdot \mathbf{v} = 0$$

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Momentum conservation

With the addition of the Lorentz force, we note

$$\begin{aligned}\frac{1}{c} \mathbf{J} \times \mathbf{B} &= \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B} = \frac{1}{4\pi} \mathbf{B} \cdot \nabla \mathbf{B} - \nabla \left(\frac{B^2}{8\pi} \right) \\ &= \nabla \cdot \left(\frac{\mathbf{B}\mathbf{B}}{4\pi} - \frac{B^2}{8\pi} \mathbf{I} \right) \quad \begin{array}{l} \text{magnetic} \\ \text{tension} \end{array} \quad \begin{array}{l} \text{magnetic} \\ \text{pressure} \end{array}\end{aligned}$$

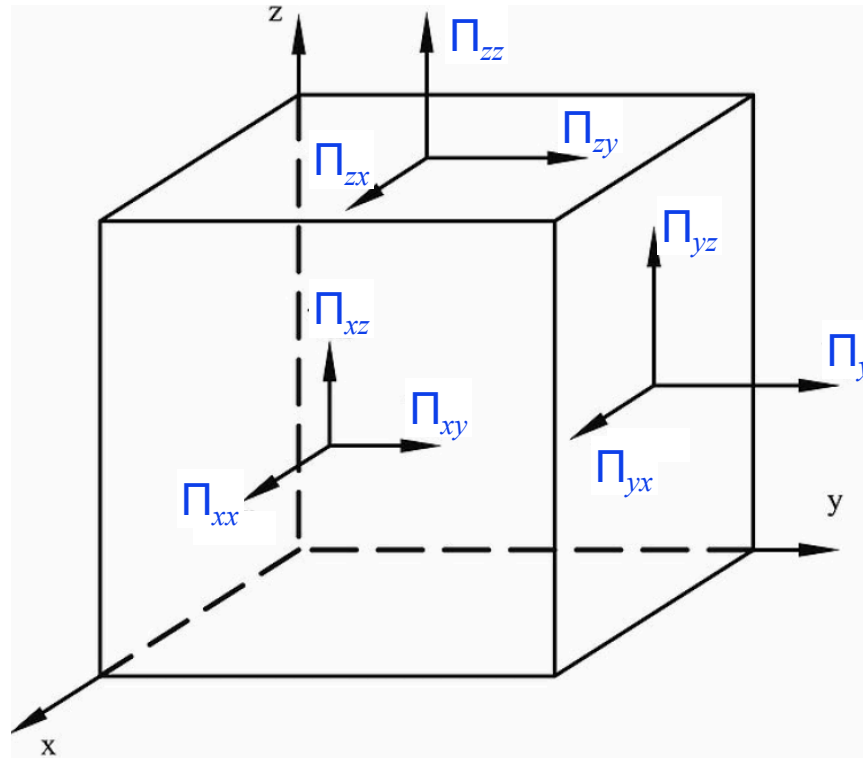
Momentum conservation becomes

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v} + \mathbf{\Pi}) = \rho \mathbf{g}$$

where the **stress tensor** is $\mathbf{\Pi} \equiv \left(P + \frac{B^2}{8\pi} \right) \mathbf{I} - \frac{\mathbf{B}\mathbf{B}}{4\pi}$
total pressure

One may further include viscous stress introduced earlier, and gravitational stress (for self-gravitating system) depending on application.

Understanding the stress tensor



- Imagine you have an infinitesimally small box.
- Forces are exerted on each face from the outside volume.
- The forces on each side have components in 3 directions.
- The stress tensor then includes all 9 quantities needed to describe these forces.

More on magnetic tension and pressure

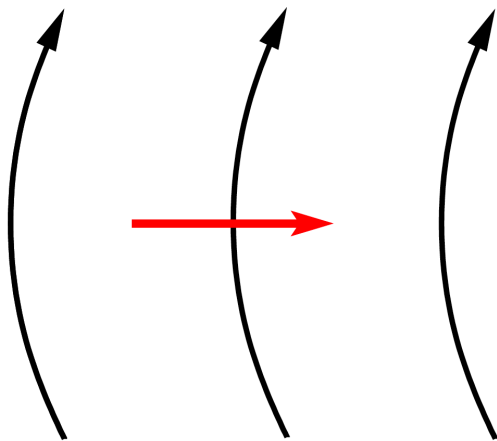
The Lorentz force is perpendicular to \mathbf{B} , but magnetic pressure sounds like an isotropic pressure. A better way to decompose the Lorentz force is as follows.

$$\frac{\mathbf{J} \times \mathbf{B}}{c} = \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi} = \frac{\mathbf{B} \cdot \nabla \mathbf{B}}{4\pi} - \nabla \left(\frac{B^2}{8\pi} \right)$$

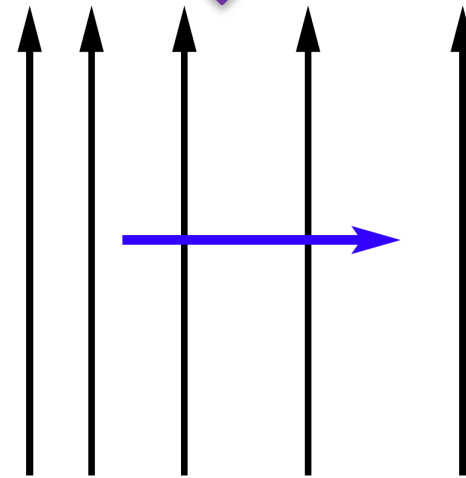
Curvature:

$$\kappa \equiv \hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}} = -\frac{\mathbf{R}}{R^2}$$

$$= \kappa \frac{B^2}{4\pi} - \nabla_{\perp} \left(\frac{B^2}{8\pi} \right)$$



Magnetic tension



Magnetic pressure

Magnetic flux conservation

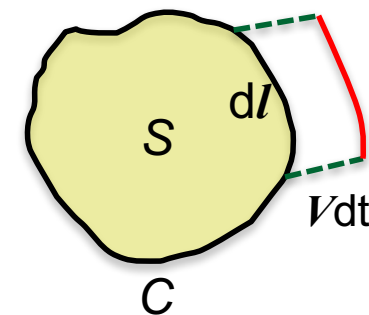
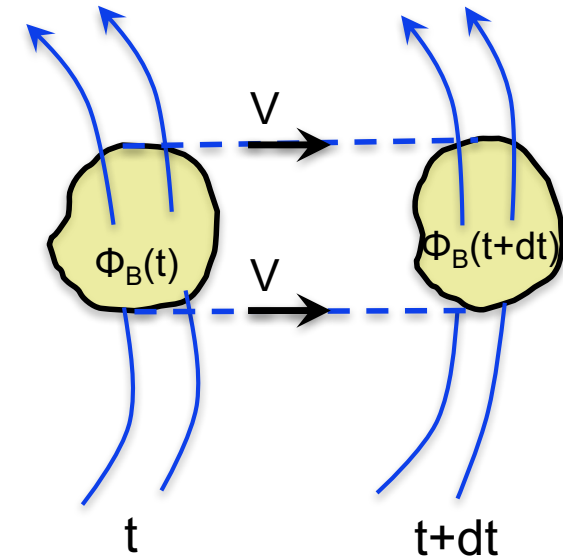
The induction equation: $\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$

already implies B flux conservation. More importantly, it implies:

The B flux through a co-moving fluid loop is constant (known as *Alfvén's theorem*).

Proof: $\Phi = \int_S \mathbf{B} \cdot d\mathbf{S}$

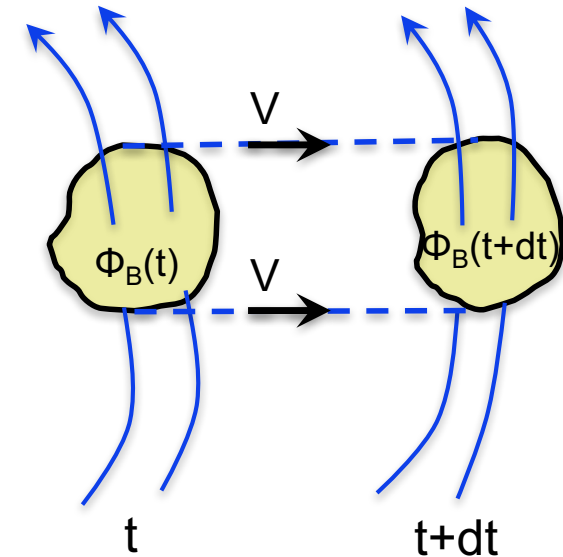
$$\begin{aligned} \frac{D\Phi}{Dt} &= \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \oint_C \mathbf{B} \cdot (\mathbf{V} \times d\mathbf{l}) \\ &= -c \int_S \nabla \times \mathbf{E} \cdot d\mathbf{S} - \oint_C (\mathbf{V} \times \mathbf{B}) \cdot d\mathbf{l} \\ &= -c \oint_C \mathbf{E} \cdot d\mathbf{l} + c \oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \end{aligned}$$



Flux freezing: physical meaning

In ideal MHD, the magnetic field and plasma are “*frozen-in*” to each other.

- The plasma cannot move across B field lines.
- If two plasma elements are initially connected by a field line, they will remain connected.
- Magnetic topology is preserved in ideal MHD.



Physically: charged particles are tied to field lines as they gyrate.

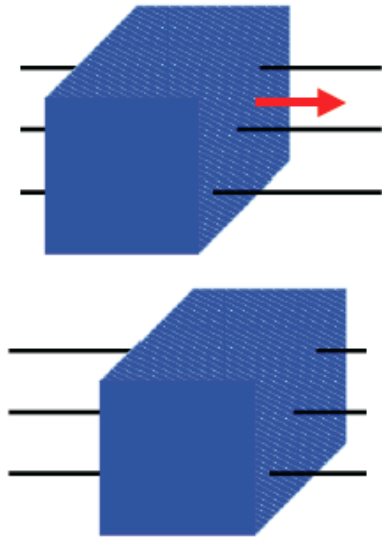
The frozen-in condition can break in non-ideal circumstances.

In particular, *magnetic reconnection* is a process that breaks magnetic field topology. It generally involves kinetic effects beyond MHD (in collisionless plasmas) and/or dissipation by resistivity.

We will briefly address this later.

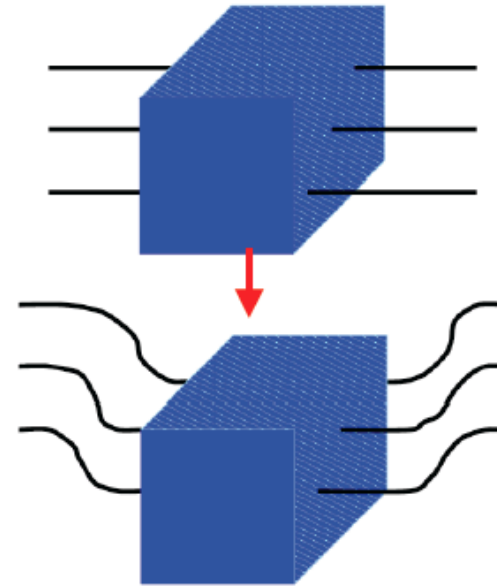
Flux freezing with weak/strong field

Strong field: matter move along field lines (beads on a wire).



$$\frac{|B|^2}{8\pi} \gg P_{\text{gas}} + \rho|\mathbf{v}|^2$$

Weak field: field lines are forced to move with the gas.



$$\frac{|B|^2}{8\pi} \ll P_{\text{gas}} + \rho|\mathbf{v}|^2$$

Strength of the B field is commonly characterized by the **plasma β** parameter:

$$\beta \equiv \frac{P_{\text{gas}}}{P_{\text{mag}}} = \frac{8\pi P_{\text{gas}}}{B^2}$$

Energy conservation

Now we re-derive energy conservation incorporating B field. New components are:

$$\rho \frac{D}{Dt} \left(\frac{v^2}{2} \right) = \rho \mathbf{v} \cdot \frac{D\mathbf{v}}{Dt} = -\rho \mathbf{v} \cdot \nabla \Phi - \mathbf{v} \cdot \nabla P + \frac{1}{4\pi} \mathbf{v} \cdot [(\nabla \times \mathbf{B}) \times \mathbf{B}]$$

$$\text{where } \frac{1}{4\pi} \mathbf{v} \cdot [(\nabla \times \mathbf{B}) \times \mathbf{B}] = \frac{1}{4\pi} (\nabla \times \mathbf{B}) \cdot (-\mathbf{v} \times \mathbf{B}) = \frac{c}{4\pi} (\nabla \times \mathbf{B}) \cdot \mathbf{E}$$

$$\frac{\partial}{\partial t} \left(\frac{B^2}{8\pi} \right) = \frac{\mathbf{B}}{4\pi} \cdot \frac{\partial \mathbf{B}}{\partial t} = \frac{c}{4\pi} \mathbf{B} \cdot (\nabla \times \mathbf{E})$$

$$\text{From vector calculus: } \nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

Therefore, the equation of energy conservation becomes:

$$\frac{\partial}{\partial t} \left[\rho \left(\frac{1}{2} v^2 + \Phi + \epsilon \right) + \frac{B^2}{8\pi} \right] + \nabla \cdot \left[\rho \mathbf{v} \left(\frac{1}{2} v^2 + \Phi + \epsilon \right) + P \mathbf{v} + \frac{c \mathbf{E} \times \mathbf{B}}{4\pi} \right] = 0$$

energy density
of the B field

Poynting flux

Energy conservation

We can rewrite the energy conservation equation

$$\frac{\partial}{\partial t} \left[\rho \left(\frac{1}{2} v^2 + \Phi + \epsilon \right) + \frac{B^2}{8\pi} \right] + \nabla \cdot \left[\rho \mathbf{v} \left(\frac{1}{2} v^2 + \Phi + \epsilon \right) + P \mathbf{v} + \frac{c \mathbf{E} \times \mathbf{B}}{4\pi} \right] = 0$$

Using $\mathbf{E} = -(\mathbf{v}/c) \times \mathbf{B}$, we arrive at:

$$\frac{\partial e}{\partial t} + \nabla \cdot (e \mathbf{v} + \Pi \cdot \mathbf{v}) = 0$$

It has exactly the same form as HD energy equation except that we have updated

the energy density: $e \equiv \rho \left(\frac{1}{2} v^2 + \Phi + \epsilon \right) + \frac{B^2}{8\pi}$

and the stress tensor: $\Pi \equiv \left(P + \frac{B^2}{8\pi} \right) \mathbf{I} - \frac{\mathbf{B}\mathbf{B}}{4\pi}$

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Generalized Ohm's law

We relax the assumption of infinite conductivity, which helps address the conditions under which ideal MHD fails.

The plasma is generally made of electrons and ions. **ions** carry almost all the mass, **representing the bulk plasmas**. Here separate out the electrons (treated as a fluid):

$$n_e m_e \frac{D\mathbf{v}_e}{Dt} = -\nabla P_e - en_e \left(\mathbf{E} + \frac{\mathbf{v}_e}{c} \times \mathbf{B} \right) - n_e m_e \frac{\mathbf{v}_e - \mathbf{v}}{\tau_c}$$

inertia pressure gradient Lorentz force “collision” with the bulk plasma

Being the lightest, e^- almost instantly respond to EM fields (to avoid huge acceleration)
=> **ignore inertia**



The balance among the other terms determine the E field.

Generalized Ohm's law

Electric field is found to be:

$$\begin{aligned}\mathbf{E} &= -\frac{\nabla P_e}{en_e} - \frac{\mathbf{v}_e}{c} \times \mathbf{B} - m_e \frac{\mathbf{v}_e - \mathbf{v}}{e\tau_c} \\ &= -\frac{\nabla P_e}{en_e} - \frac{\mathbf{v}}{c} \times \mathbf{B} - \frac{\mathbf{v}_e - \mathbf{v}}{c} \times \mathbf{B} - m_e \frac{\mathbf{v}_e - \mathbf{v}}{e\tau_c}\end{aligned}$$

Note: $\mathbf{J} = -en_e(\mathbf{v}_e - \mathbf{v}_i)$

We arrive at the [generalized Ohm's law](#):

$$\mathbf{E} = -\frac{\mathbf{v}}{c} \times \mathbf{B} - \frac{\nabla P_e}{en_e} + \frac{1}{en_e c} \mathbf{J} \times \mathbf{B} + \frac{m_e}{e^2 n_e \tau_c} \mathbf{J}$$

e^- pressure gradientHall termresistivity

Generalized Ohm's law

Are additional terms in the **generalized Ohm's law** important?

$$\mathbf{E} = -\frac{\mathbf{v}}{c} \times \mathbf{B} - \frac{\nabla P_e}{en_e} + \frac{1}{en_e c} \mathbf{J} \times \mathbf{B} + \frac{m_e}{e^2 n_e \tau_c} \mathbf{J}$$

One can show that these two terms are relevant only at microscopic scales:

$$d_i \sim \frac{c}{\omega_{pi}} = \sqrt{\frac{m_i c^2}{4\pi e^2 n_i}} \approx 230 \text{ km} \left(\frac{n_i}{\text{cm}^{-3}} \right) \quad (\text{ion inertial length})$$

While we have ignored electron inertia, one can also show that that term is relevant at even smaller scales:

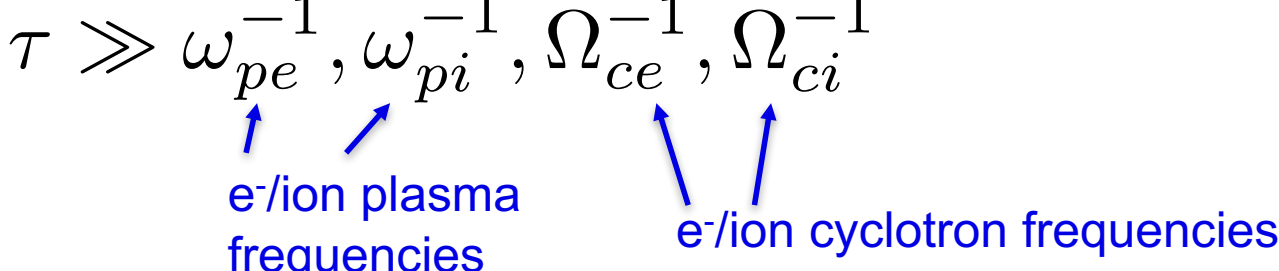
$$d_e \sim \frac{c}{\omega_{pe}} = \sqrt{\frac{m_e c^2}{4\pi e^2 n_e}} \approx 5.3 \text{ km} \left(\frac{n_e}{\text{cm}^{-3}} \right) \quad (\text{electron inertial length})$$

This gives *one example* that **MHD applies only on macroscopic scales.**

Applicability and limitations of MHD

- MHD applies to timescales much longer than

$$\tau \gg \omega_{pe}^{-1}, \omega_{pi}^{-1}, \Omega_{ce}^{-1}, \Omega_{ci}^{-1}$$



e-/ion plasma frequencies e-/ion cyclotron frequencies

- MHD applies to length scales much larger than

$$L \gg \lambda_D, r_{Le}, r_{Li}, c/\omega_{pe}, c/\omega_{pi}$$



Debye length e-/ion Larmor radii e-/ion inertial lengths

- MHD requires the particle distribution function to be (at least approximately) **isotropic** and **Maxwellian** with $T_i = T_e$.

Resistive MHD

We are left with the **standard Ohm's law**: $\mathbf{J} = \sigma \mathbf{E}' = \sigma \left(\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right)$

where electric conductivity $\sigma \equiv \frac{e^2 n_e \tau_c}{m_e}$.

The induction equation now reads

$$\frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \frac{c}{\sigma} \nabla \times \mathbf{J}$$

which finally becomes

$$-\frac{c^2}{4\pi\sigma} \nabla \times (\nabla \times \mathbf{B})$$

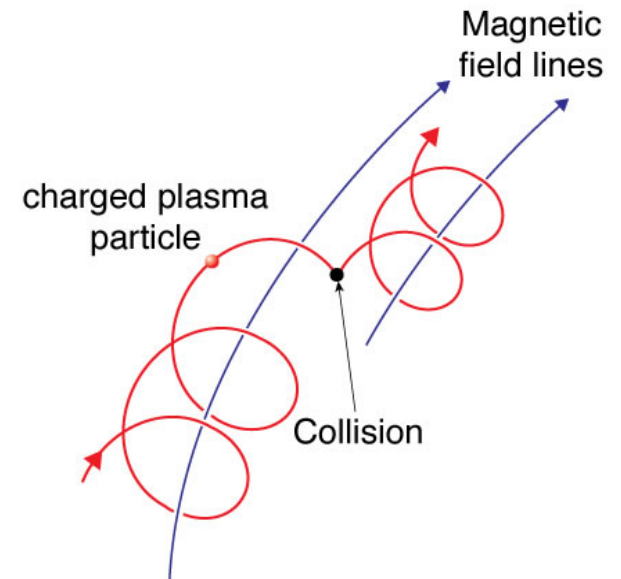
$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

where **Ohmic resistivity** is given by $\eta \equiv \frac{c^2}{4\pi\sigma} = \frac{m_e c^2}{4\pi e^2 n_e \tau_c}$

Resistive MHD

- Resistivity **breaks the frozen-in condition**, allowing field lines to slide through the plasma.

Physically, this is due to **particle collisions**:



- Resistivity **leads to energy dissipation (Ohmic heating)**.

Energy conservation becomes:

$$\frac{\partial e}{\partial t} + \nabla \cdot \left(e\mathbf{v} + \Pi \cdot \mathbf{v} + \frac{c\mathbf{E}_O \times \mathbf{B}}{4\pi} \right) = 0 \quad \text{where } \mathbf{E}_O \equiv \frac{4\pi\eta}{c^2} \mathbf{J}$$

with the associated Ohmic heating: $\rho T \frac{Ds}{Dt} = \mathbf{E}_O \cdot \mathbf{J}$

When is resistivity important?

Order of magnitudes from the induction equation:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

$$\sim B/T$$

$$\sim BV/L$$

$$\sim \eta B/L^2$$

Define the **magnetic Reynolds number**:

$$\text{Re}_M \equiv \frac{VL}{\eta} \quad \text{resistivity is dominant when } \text{Re}_M \sim 1 \text{ or less.}$$

In most astrophysical systems, this number is **huge**:

In the solar interior, $\text{Re}_M \sim 10^{16}$

In a 1pc parcel of the ISM, $\text{Re}_M \sim 10^{18}$



Flux freezing is valid.

Even with large Re_M for the **bulk flow**, resistivity can still play important roles at small scales (e.g., reconnection, discontinuities, turbulent dissipation).

Kinetic vs. MHD

- MHD is a low-frequency, long-wavelength approximation, and requires particle distribution function to be approximately Maxwellian.
 - MHD fails when kinetic effects are important, and when the distribution function deviates substantially from Maxwellian.
 - Dissipation which involves small-scale microphysics (e.g., magnetic reconnection, turbulent dissipation, many micro-scale instabilities).
 - Dynamics of cosmic-rays (DF is \sim power-law)
-
- ◆ MHD is reasonable for many astrophysical plasmas at large scales.
 - ◆ Extensions are often needed (e.g., energy transport, thermal conduction, viscosity/resistivity).
 - ◆ For many systems, MHD is applicable for certain physical problems, whereas others require kinetic description.

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MHD waves

We consider perturbations on top of a static homogeneous plasma with uniform field \mathbf{B}_0 , and start with linearized MHD equations (subscript 1 for perturbed quantities).

We then decompose all perturbed quantities into Fourier modes in the form of $e^{i(\omega t - \mathbf{k} \cdot \mathbf{x})}$ to derive the dispersion relation.

$$\begin{aligned} \frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}_1 &= 0, \\ \rho_0 \frac{\partial \mathbf{v}_1}{\partial t} &= -\nabla P_1 + \frac{(\nabla \times \mathbf{B}_1) \times \mathbf{B}_0}{4\pi}, \\ \frac{\partial \mathbf{B}_1}{\partial t} &= \nabla \times (\mathbf{v}_1 \times \mathbf{B}_0), \\ \frac{\partial P_1}{\partial t} + \gamma P_0 \nabla \cdot \mathbf{v}_1 &= 0 \end{aligned}$$

Independent from others (in adiabatic MHD) => entropy wave

→

$$\begin{aligned} -\omega \rho_1 + \rho_0 \mathbf{k} \cdot \mathbf{v}_1 &= 0 \\ -\omega \rho_0 \mathbf{v}_1 &= -P_1 \mathbf{k} + \frac{(\mathbf{k} \times \mathbf{B}_1) \times \mathbf{B}_0}{4\pi}, \\ -\omega \mathbf{B}_1 &= \mathbf{k} \times (\mathbf{v}_1 \times \mathbf{B}_0), \\ \omega P_1 + \gamma P_0 \mathbf{k} \cdot \mathbf{v}_1 &= 0 \end{aligned}$$

Total of 8 equations (adiabatic MHD), with 7 degrees of freedom (since $\nabla \cdot \mathbf{B}_1 = 0$):
 7 waves (1+2x3). For isothermal MHD, there are 6 waves (2x3).

MHD waves

Without loss of generality, may take \mathbf{B}_0 to be along the z direction, and let the angle between \mathbf{k} and z to be θ .

$$-\omega\rho_0\mathbf{v}_1 = -P_1\mathbf{k} + \frac{(\mathbf{k} \times \mathbf{B}_1) \times \mathbf{B}_0}{4\pi},$$

$$-\omega\mathbf{B}_1 = \mathbf{k} \times (\mathbf{v}_1 \times \mathbf{B}_0),$$

$$\omega P_1 + \gamma P_0 \mathbf{k} \cdot \mathbf{v}_1 = 0$$

After some algebra, we obtain

$$-\omega^2\mathbf{v}_1 = -c_s^2(\mathbf{k} \cdot \mathbf{v}_1)\mathbf{k} - v_A^2\{\mathbf{k} \times [\mathbf{k} \times (\mathbf{v}_1 \times \mathbf{e}_z)]\} \times \mathbf{e}_z$$

where $c_s = \sqrt{\gamma P_0/\rho_0}$ is the adiabatic sound speed as usual.

$v_A \equiv \frac{B_0}{\sqrt{4\pi\rho_0}}$ is the **Alfvén speed** whose meaning will become clear shortly.

This is a linear equation for \mathbf{v}_1 , whose solution gives the **dispersion relation**.

Alfvén waves

The linear equations permit **an incompressible mode** which is unique in MHD.

$$-\omega^2 \mathbf{v}_1 = -c_s^2 (\mathbf{k} \cdot \mathbf{v}_1) \mathbf{k} - v_A^2 \{ \mathbf{k} \times [\mathbf{k} \times (\mathbf{v}_1 \times \mathbf{e}_z)] \} \times \mathbf{e}_z$$

For incompressible mode, $\mathbf{k} \cdot \mathbf{v}_1 = 0$

From the above, one can further find: $\mathbf{v}_1 \cdot \mathbf{e}_z = 0$

The above equation reduces to:

$$-\omega^2 \mathbf{v}_1 = -k_z^2 v_A^2 \mathbf{v}_1$$

acceleration restoring force:

Lorentz force (magnetic tension)

This gives the dispersion relation for **Alfvén waves**:

$$\omega = \pm k_z v_A$$

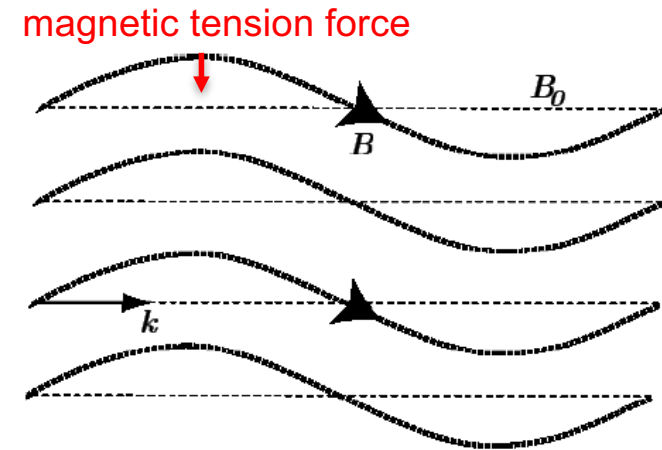


Hannes Alfvén
Nobel prize (1970)

Alfvén waves: basic properties

It is a **transverse wave** $\begin{pmatrix} \mathbf{k} \cdot \mathbf{v}_1 = 0 \\ \mathbf{k} \cdot \mathbf{B}_1 = 0 \end{pmatrix}$.

The physics of the Alfvén waves is analogous to **waves on a string**:



There is an **equipartition of kinetic energy and magnetic energy** in the wave:
$$\frac{B_1^2}{8\pi} = \frac{B_0^2 k_z^2}{8\pi\omega^2} v_1^2 = \frac{1}{2} \rho_0 v_1^2$$

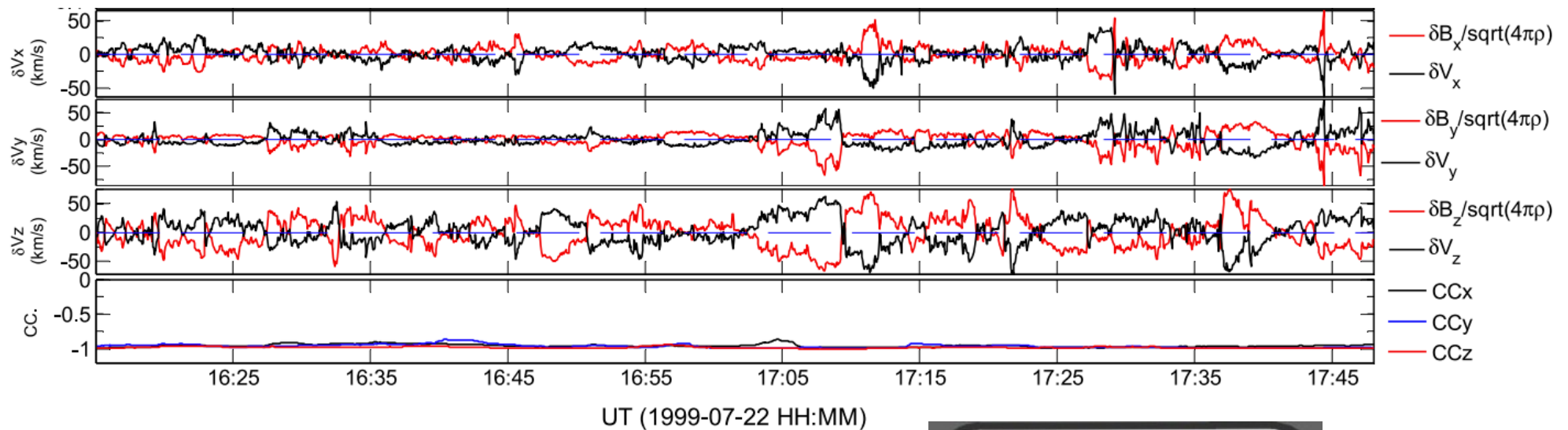
Dispersion relation can be rewritten as: $\omega = \pm \mathbf{k} \cdot \mathbf{v}_A$ ← along \mathbf{B}_0 .

Phase velocity: $V_p = \pm \frac{\omega}{k} \hat{\mathbf{k}} = (v_A \cos \theta) \hat{\mathbf{k}}$

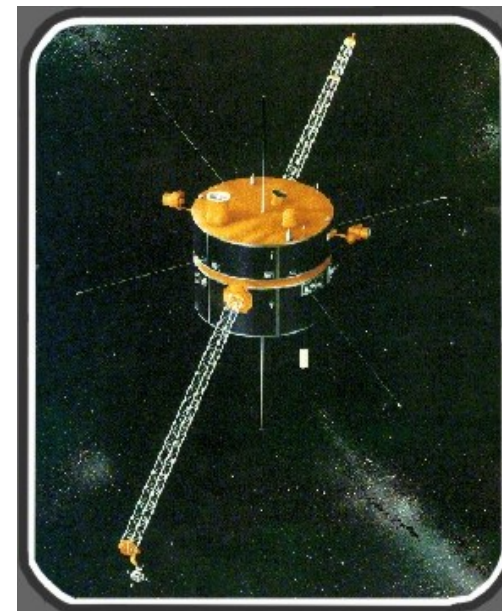
Group velocity: $V_g = \frac{\partial \omega}{\partial \mathbf{k}} = \mathbf{v}_A$ Energy propagates along field lines.

In situ measurement of Alfvén waves

Shi et al. 2015



Identification of **anti-correlated perturbations in velocity and magnetic fields (as Alfvén waves)** in the solar wind at ~ 1 AU from the WIND spacecraft data.



WIND spacecraft

Magnetosonic waves

There are **two compressible modes** with a dispersion relation:

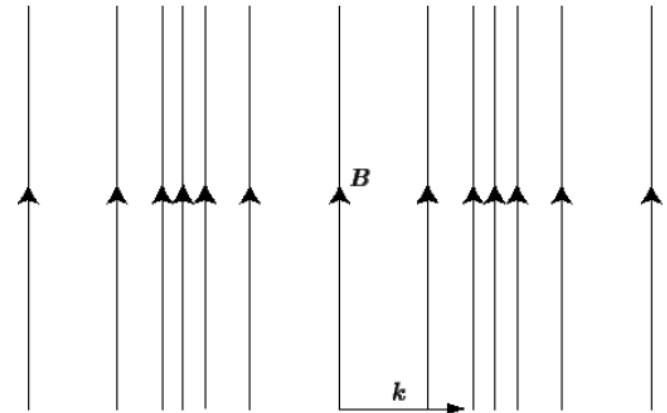
$$\frac{\omega^2}{k^2} = \frac{(c_s^2 + v_A^2)}{2} \pm \left[\left(\frac{c_s^2 + v_A^2}{2} \right)^2 - c_s^2 v_A^2 \cos^2 \theta \right]^{1/2}$$

They are known as **fast (+) and slow (-) magnetosonic waves**.

Analogous to sound waves modified by a B field.

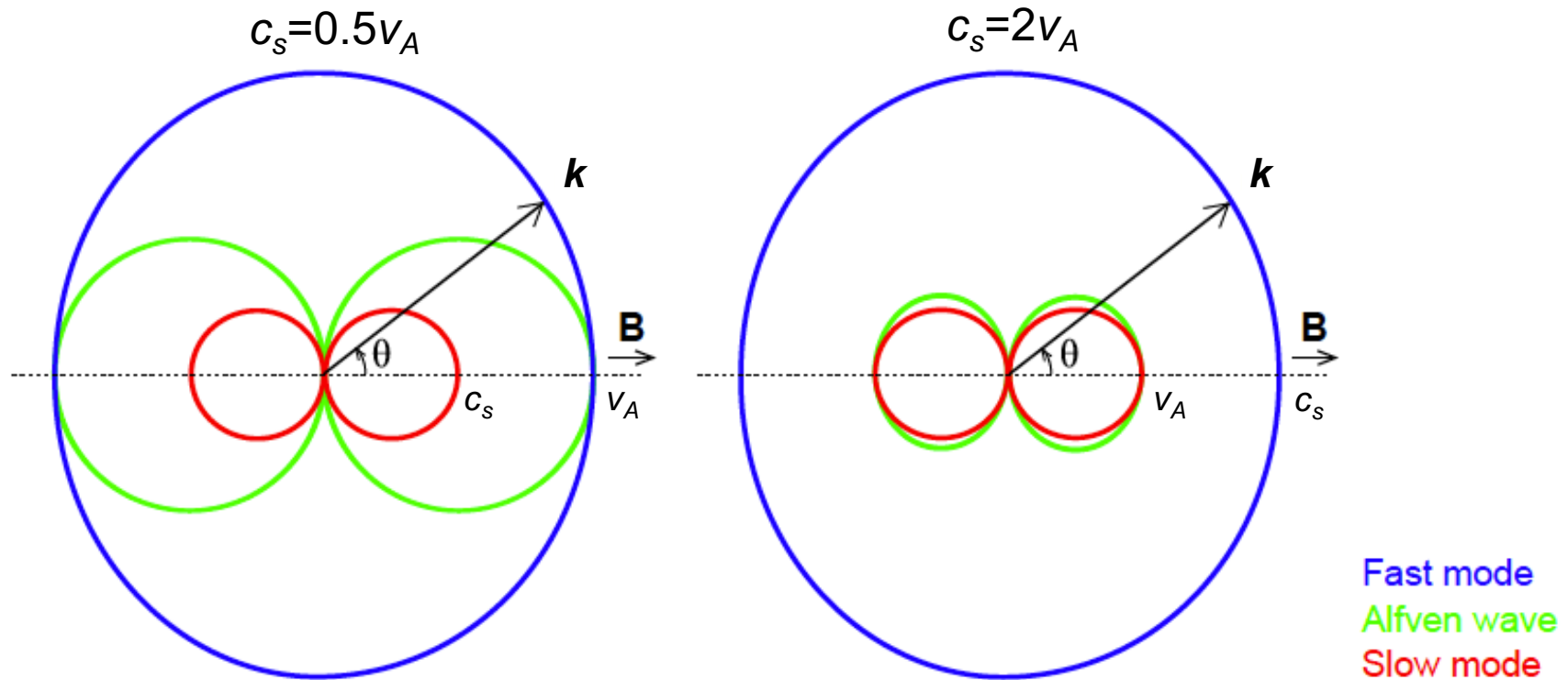
The restoring force includes contributions from both thermal and magnetic pressure. Roughly speaking:

- In a slow wave, the two effects are **out of phase**.
- In a fast wave, the two effects are **in phase**.



The general behaviors are complex and are better visualized in **Friedrichs diagrams**.

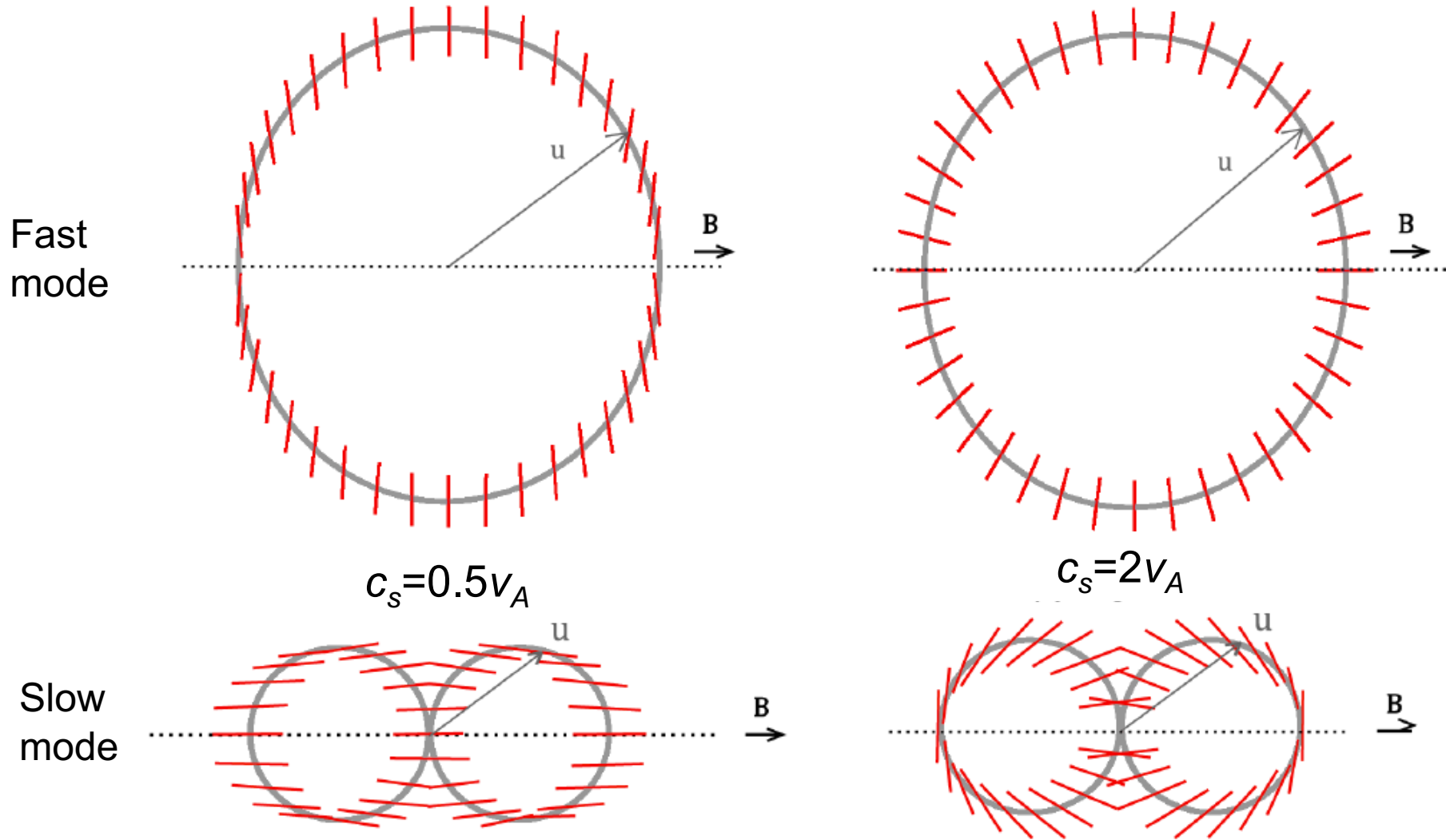
Friedrichs diagram



The slow wave can not propagate orthogonally.
The fast wave propagate quasi-isotropically.

Directions of fluid displacements

Spruit, 2013



Outline

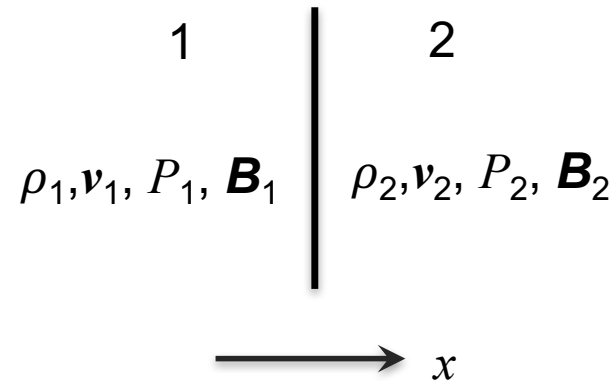
- Astrophysical fluids as plasmas
- The MHD formulation
- Conservation laws and physical interpretation
- Generalized Ohm's law, and limitations of MHD
- MHD waves
- **MHD shocks and discontinuities**
- MHD instabilities (examples)

Discontinuities in MHD: jump conditions

Mass conservation: $[\rho v_x]_1^2 = 0$

Momentum conservation:

$$\left[\rho \mathbf{v} v_x + \left(P + \frac{B^2}{8\pi} \right) \mathbf{e}_x - \frac{\mathbf{B}}{4\pi} B_x \right]_1^2 = 0$$



Energy conservation:

$$\left[\left(\frac{1}{2} \rho v^2 + \frac{\gamma P}{\gamma - 1} + \frac{B^2}{4\pi} \right) v_x - \frac{\mathbf{B} \cdot \mathbf{v}}{4\pi} B_x \right]_1^2 = 0$$

Divergence-free constraint: $[B_x]_1^2 = 0$

Tangential E field must be continuous: $[(\mathbf{v} \times \mathbf{B}) \times \mathbf{e}_x]_1^2 = 0$

MHD shocks

The jump conditions contain rich physics that are related to different MHD wave modes.

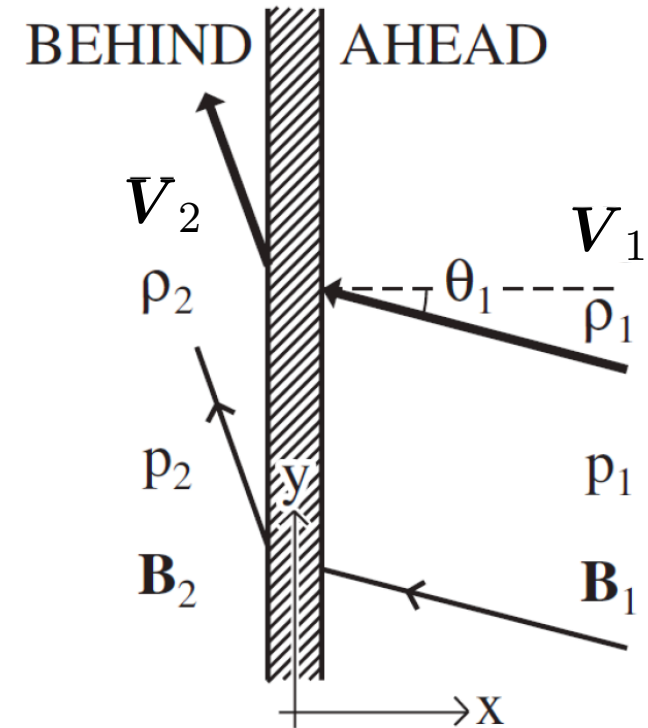
The velocity of a fast/slow mode shock must exceed the fast/slow speed in the upstream

Define θ as the angle between the upstream \mathbf{B} field and shock normal:

$\theta = 0$: parallel shock

$\theta = \pi/2$: perpendicular shock

$0 < \theta < \pi/2$: oblique shock

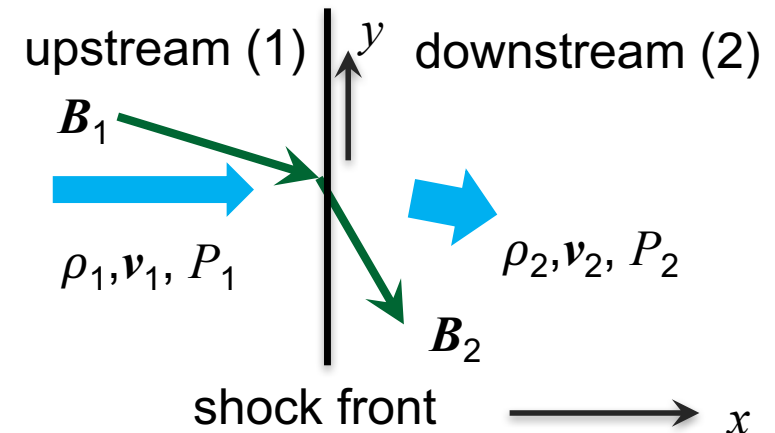


It is common to define Alfvénic Mach number: $M_A \equiv \frac{V_1}{v_A}$

Example: a strong (fast mode) shock

Strong means: shock kinetic energy \gg thermal+magnetic energy @ upstream.

Choose a frame where upstream velocity is along shock normal:



$$\rho_1 v_{1x} = \rho_2 v_{2x}$$

$$\rho_1 v_{1x}^2 = \rho_2 v_{2x}^2 + P_2 - \frac{B_2^2}{8\pi}$$

$$0 = \rho_2 v_{2y} v_{2x} - \frac{B_{2y} B_x}{4\pi}$$

$$\rho_1 v_{1x}^3 = \rho_2 v_{2x}^3 + \frac{2\gamma}{\gamma - 1} P_2 v_{2x} + \frac{B_2^2 v_{2x}}{4\pi} - \frac{B_x v_{x2} + B_{2y} v_{2y}}{4\pi} B_x$$

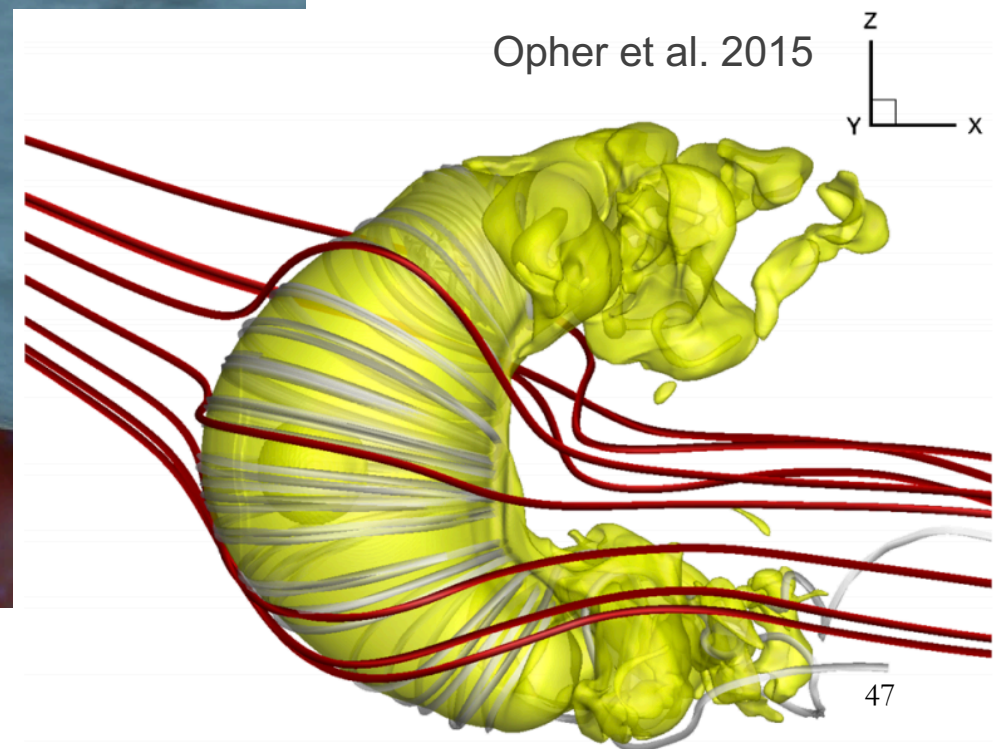
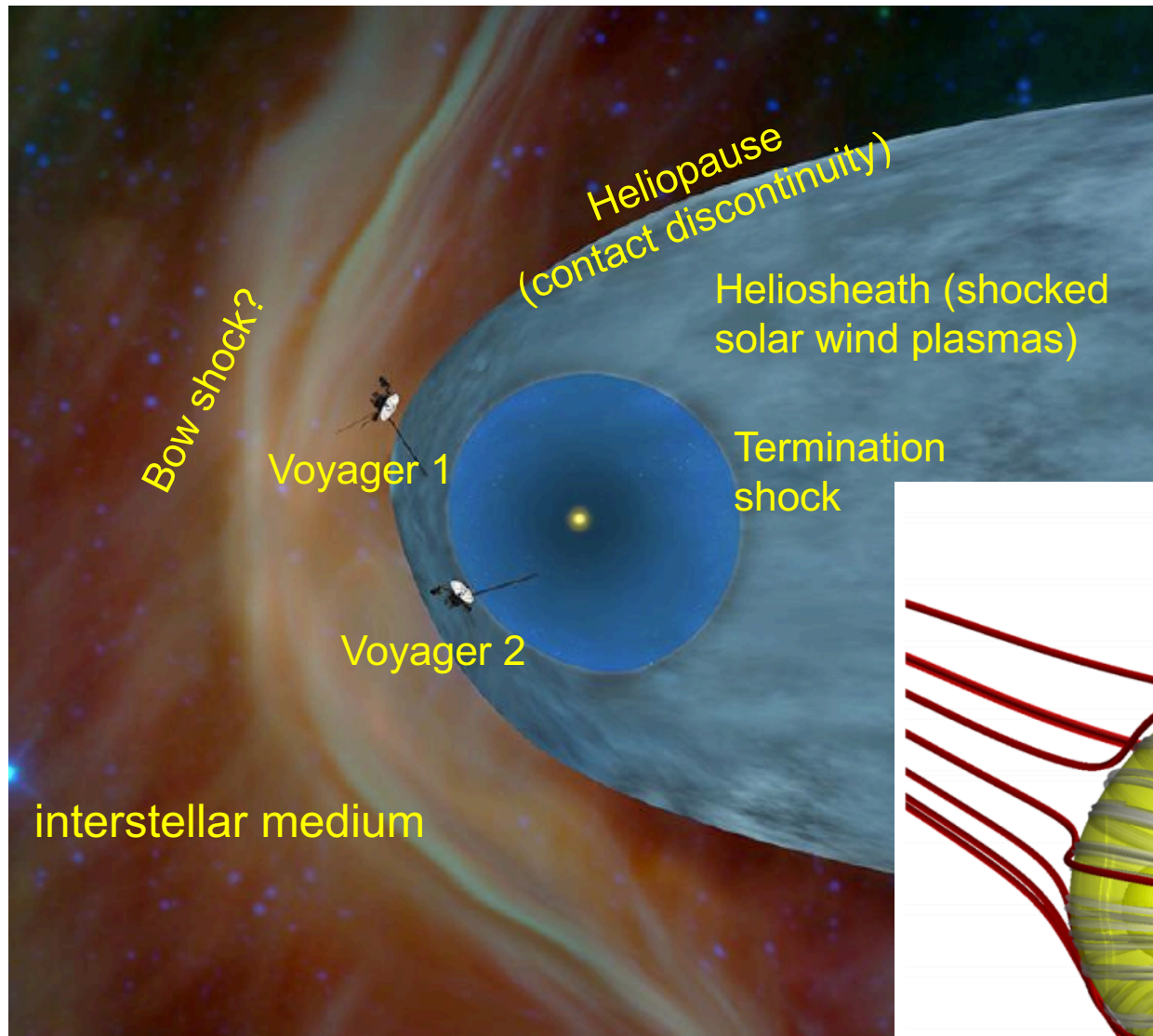
$$v_{1x} B_{1y} = v_{2x} B_{2y} - v_{2y} B_x$$

The end result is similar to a strong hydrodynamic shock:

$$\rho_2 = 4\rho_1, \quad v_{2x} = v_{1x}/4, \quad B_{2y} = 4B_{1y}, \quad v_{2y} = \frac{B_{1y}}{B_x} \frac{v_{Ax}^2}{v_{1x}} \quad (\text{for } \gamma = 5/3)$$

perpendicular B is compressed tangential velocity is developed at downstream

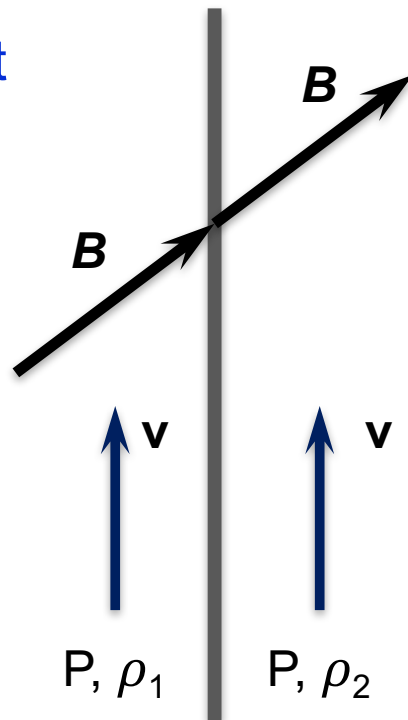
Example: shocks in the heliosphere



Contact and tangential discontinuities

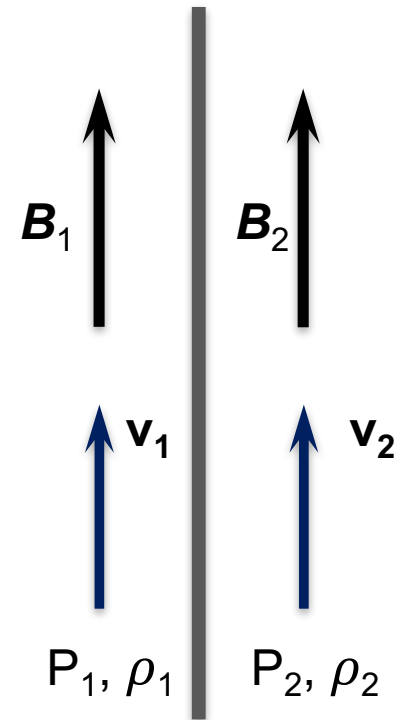
In both cases, there is no flow across the discontinuity (they are not shocks).

Contact



All quantities except density are continuous across the discontinuity.

Tangential

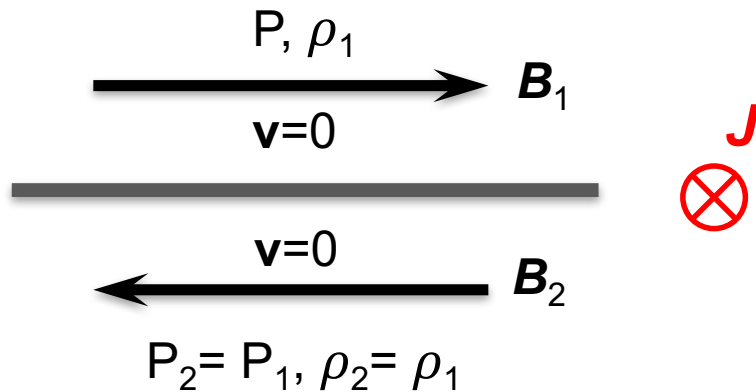


Both B and v are tangential to the discontinuity.

Total pressure is continuous across the discontinuity.

Current sheet

Current sheet is a special example of tangential discontinuity.

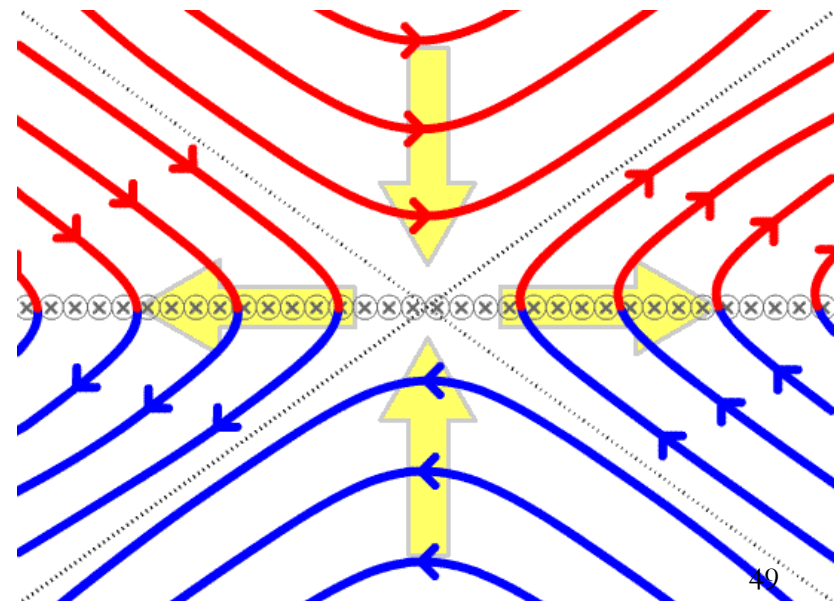


Strong current in the current sheet can lead to strong dissipation via resistivity.

The configuration is also subject to MHD instabilities (e.g., tearing mode and drift-kink).

This configuration is the prototype for studying **magnetic reconnection**.

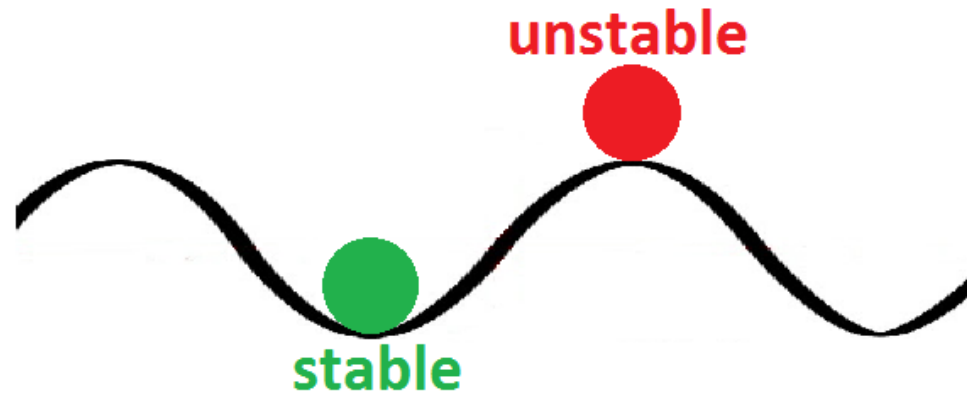
Leading to change of field topology with rapid dissipation of magnetic energy.



Outline

- Astrophysical fluids as plasmas
- The MHD formulation
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- MHD waves
- MHD shocks and discontinuities
- **MHD instabilities (examples)**

Fluid/MHD instability: guiding principles



Consider a Lagrangian perturbation with displacement vector ξ on top of an equilibrium configuration.

In response to the perturbation there is a force $F(\xi)$.

This configuration is **unstable** if $\xi \cdot F(\xi) > 0$

Namely, the force encourages the displacements, making the perturbation grow.

Fluid/MHD instability: linear analysis

In general, one needs to conduct standard **linear analysis**:

One can either adopt the **Eulerian approach** as we did before in linear waves, or the **Lagrangian approach** described by a displacement ξ that satisfies

$$\delta \mathbf{v} = \frac{\partial \xi(\mathbf{r}, t)}{\partial t}$$

With linearized equations, we seek for solutions of the form:

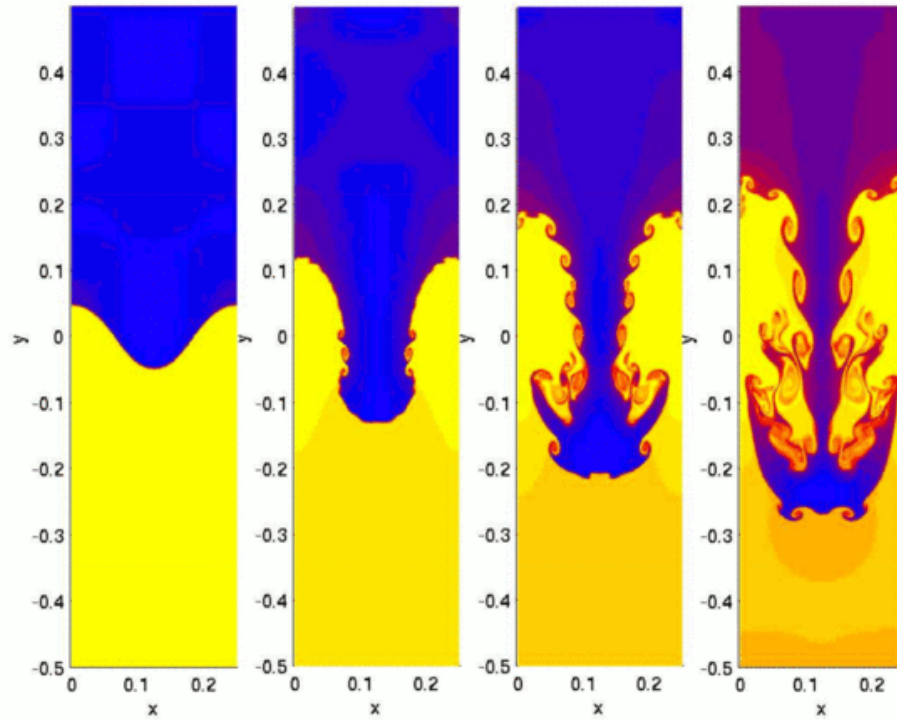
$$\xi(\mathbf{r}, t) = \xi(\mathbf{r}) e^{i\omega t}$$

↖ an eigenfunction
(e.g., a Fourier mode)

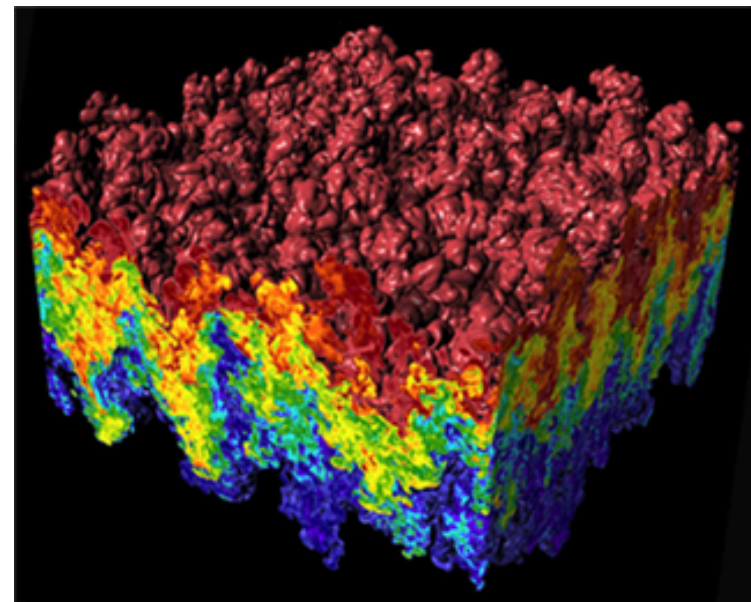
One can prove that in ideal MHD, solutions in ω^2 is always real:

- If $\omega^2 > 0$ for all solutions, then the equilibrium is **stable (oscillatory)**.
- If $\omega^2 < 0$ for some solutions, then the equilibrium is **unstable (exponential growth)**.

Rayleigh-Taylor instability



Unstable because heavy fluid is placed above light fluid.



Can lead to substantial mixing



It occurs in supernova explosions and subsequent evolution of the remnant

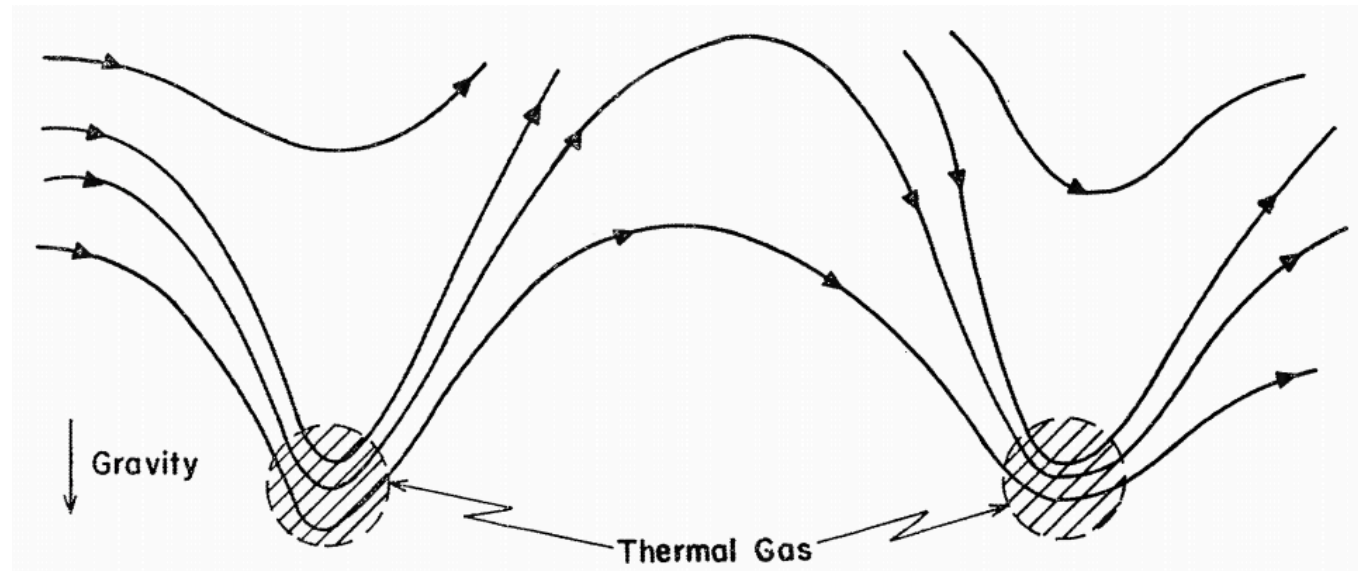
Parker instability

Parker 1966

Starting from a magneto-hydrostatic equilibrium in the galactic disk:

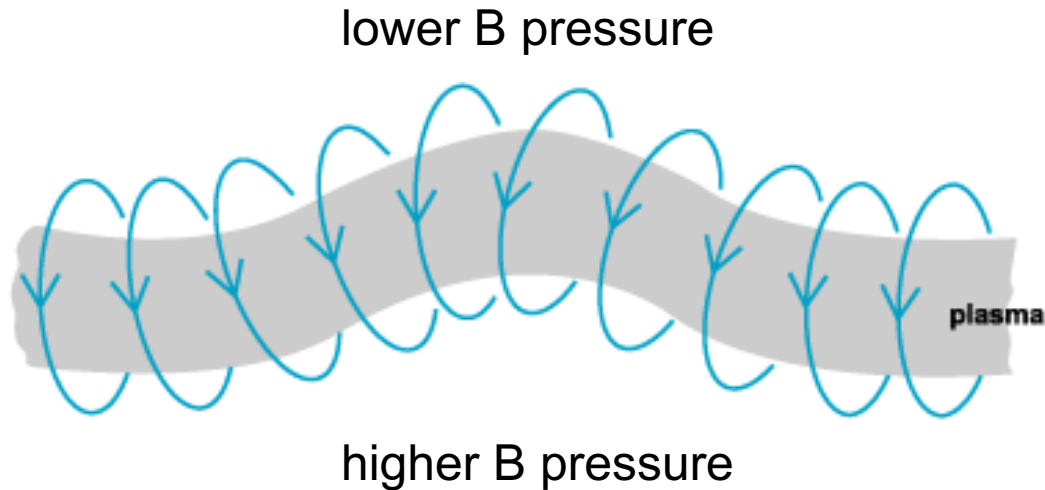
$$\frac{d}{dz} \left(\underbrace{p}_{\text{gas pressure}} + \underbrace{P}_{\text{CR pressure}} + \underbrace{\frac{B^2}{8\pi}}_{\text{B field pressure}} \right) = -\rho(z) g(z)$$

This configuration can be unstable at long wavelength due to **magnetic buoyancy**.



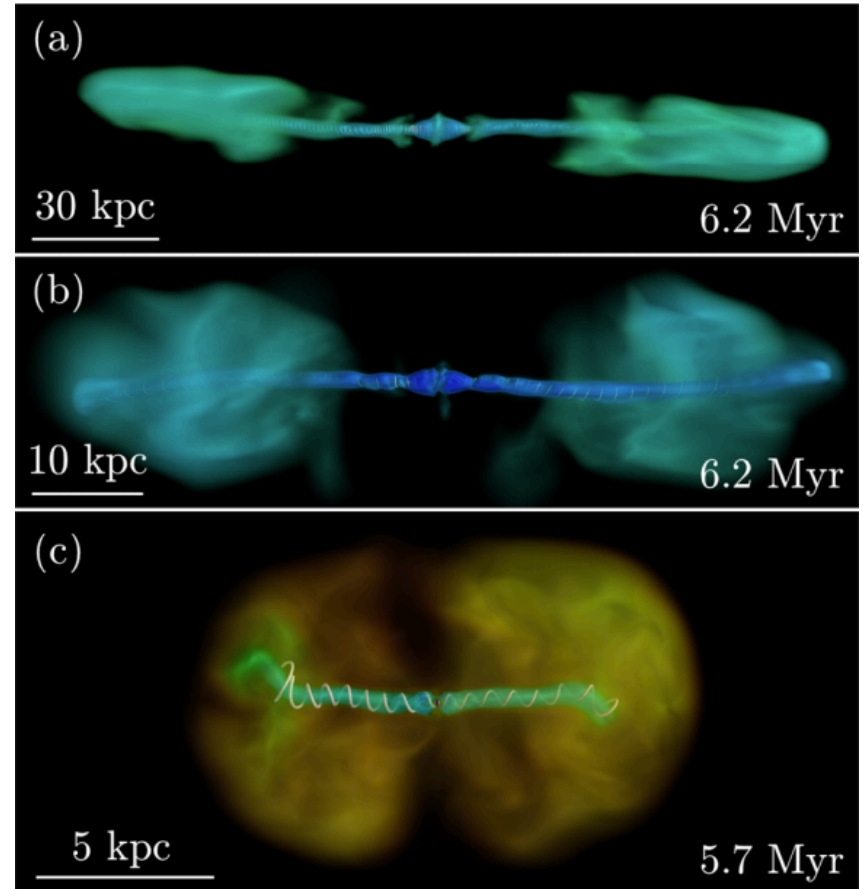
It may be responsible for the formation of molecular cloud.

Kink instability



The distortion above leads to a difference in magnetic pressure that further enhances the distortion.

Adding an axial field is stabilizing which offers a restoring force from magnetic tension.



Tchekhovskoy & Bromberg, 2016

Kink instability is closely related to the stability of astrophysical jets.

Magnetorotational instability (MRI)

- Rayleigh criterion for unmagnetized rotating disks:

Unstable if: $\frac{d(\Omega R^2)}{dR} < 0$ (Rayleigh, 1916)

Confirmed experimentally (Ji et al. 2006).

All astrophysical disks should be stable against this criterion.

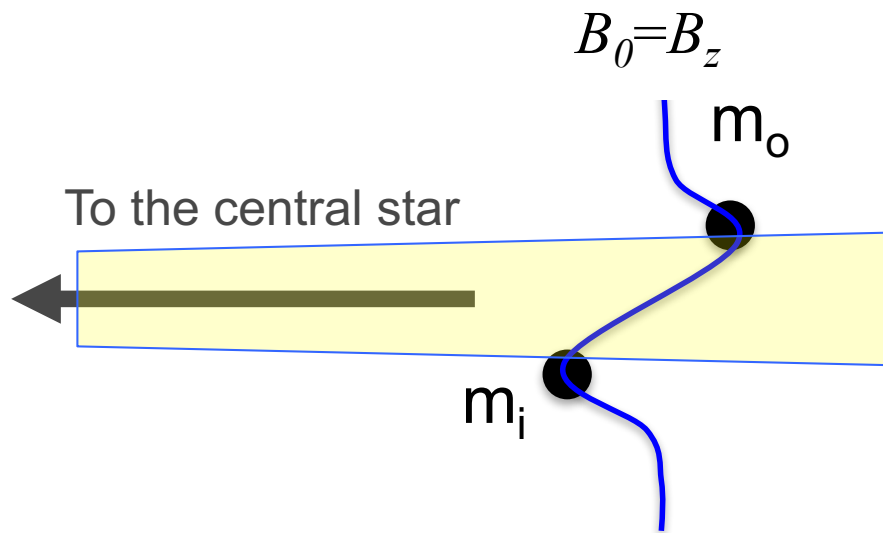
- Including (a vertical, well-coupled) magnetic field qualitatively changes the criterion (even as $B \rightarrow 0$):

Unstable if: $\frac{d\Omega}{dR} < 0$ Velikhov (1959),
Chandrasekhar (1960),
Balbus & Hawley (1991)

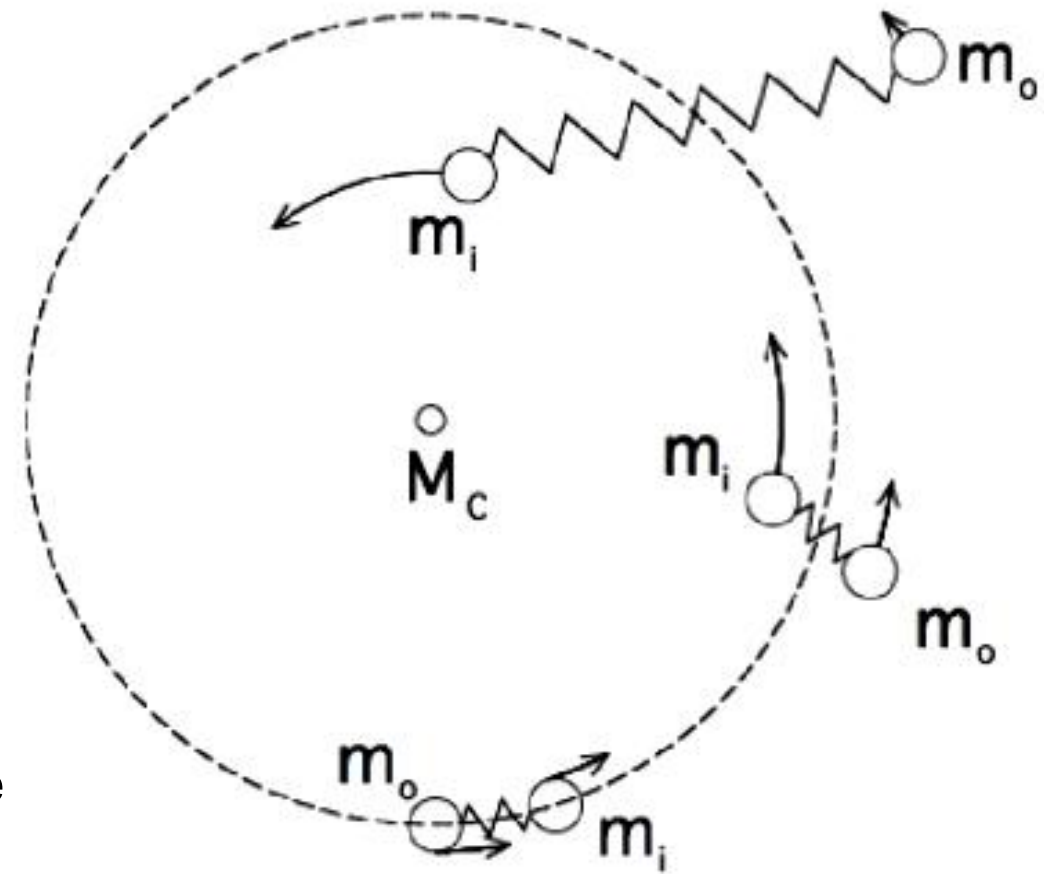
All astrophysical disks should be unstable!

Magnetorotational instability (MRI)

Edge on view:

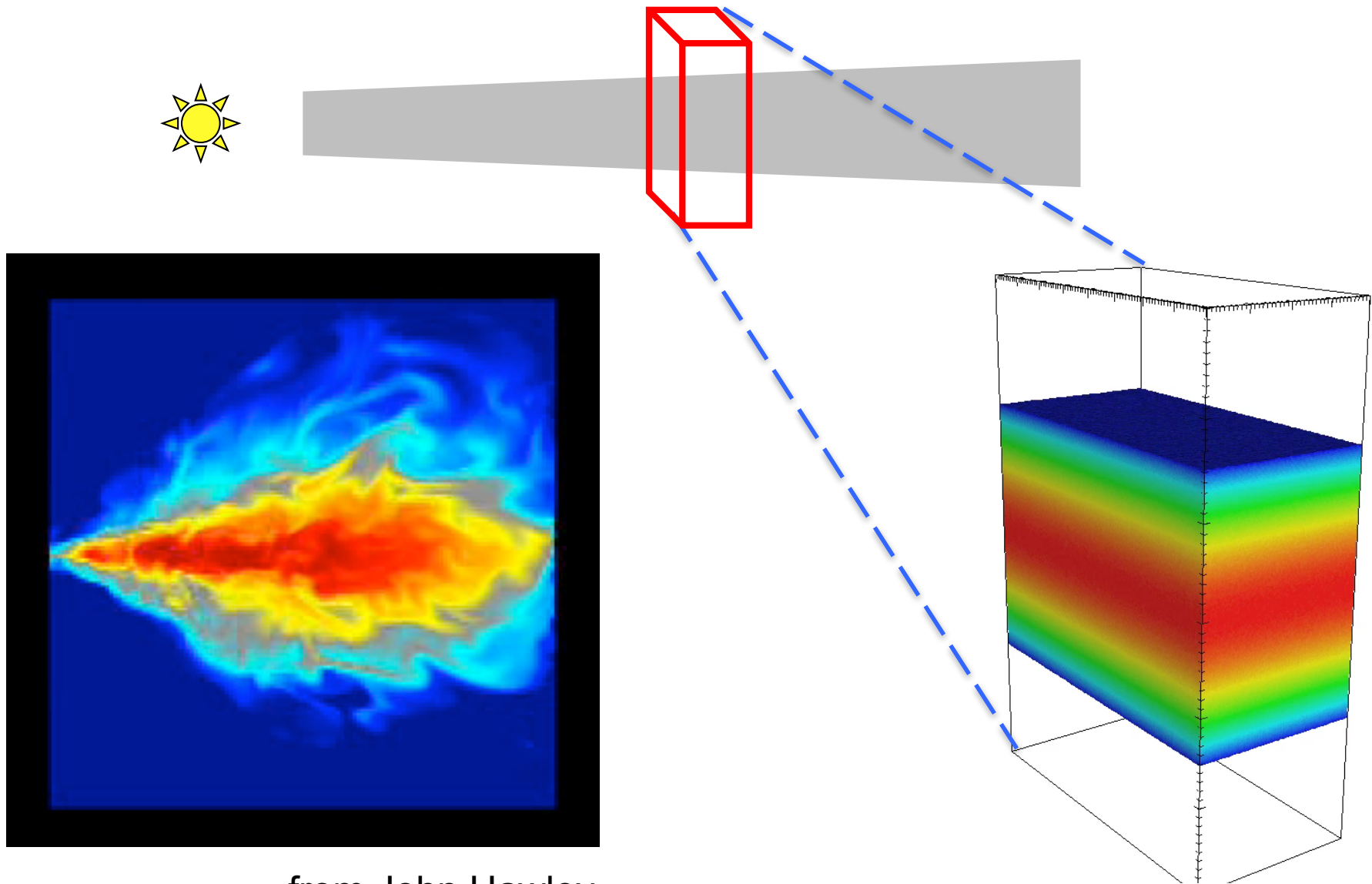


Face on view:



Magnetic tension force behaves like a spring.

Local and global simulations of the MRI



from John Hawley

Summary

- Plasmas are ubiquitous in astrophysical systems.
- MHD, the fluid description of plasmas, is a long-wavelength, low-frequency limit approximation.
- Lorentz force consists of magnetic tension and pressure.
- In ideal MHD, magnetic flux is frozen-in to the fluid, whereas resistivity breaks this condition.
- Three types of MHD waves: slow/fast magnetosonic (compressible) waves, and the Alfvén wave (incompressible).
- MHD shocks/discontinuities have much richer variety of behaviors.
- There are a wide variety of HD/MHD instabilities with substantial astrophysical significance.