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# Motion of Charged Particles

The equation of motion for charged particles with mass $m$ and charge $q$ reads

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{v} \ , \qquad \frac{d\boldsymbol{v}}{dt} = \frac{q}{m}\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}\right) \ . \tag{1}$$

**Uniform and static B field, no E field**

In this simplest situation, let $\boldsymbol{B} = B_0\hat{z}$, the equation of motion is

$$\ddot{x} = \Omega\dot{y} \ , \quad \ddot{y} = -\Omega\dot{x} \ , \quad z = \text{const} \ , \quad \text{where } \Omega \equiv \frac{qB_0}{mc} \ . \tag{2}$$

The particle orbit can be easily solved

$$x(t) = \frac{v_\perp}{\Omega}\sin(\Omega t + \alpha) + x_0 \ , \quad y(t) = \frac{v_\perp}{\Omega}\cos(\Omega t + \alpha) + y_0 \ , \quad z(t) = v_\parallel t + z_0 \ , \tag{3}$$

where $\alpha$, $x_0$, $y_0$, $z_0$, together with $v_\parallel$ and $v_\perp$ are determined from initial conditions.

The motion of the particle is thus a superposition of motion along $\boldsymbol{B}$ at constant velocity $v_\parallel$ and circular motion around the guiding center, $\boldsymbol{R}_g = (x_0, y_0, z_0 + v_\parallel t)$ at a constant velocity $v_\perp$. The radius of the circle, $r_L \equiv v_\perp/|\Omega|$, is known as the Larmor radius or gyro-radius. The phase of the circular motion, $\phi(t) \equiv \Omega t + \alpha$, is called the gyro-phase. Particle pitch angle $\theta$ is defined as the angle between particle velocity and magnetic field, and we have $\tan\theta = v_\perp/v_\parallel$. The sense of gyration is left-handed about $\boldsymbol{B}$ for ions, and is right-handed for electrons. We also emphasize that for particle gyro-motion in a constant field, the guiding center is tied to a fixed field line and can not cross to other field lines.

The discussions above are easily generalized to the relativistic regime by simply replacing $m$ by $\gamma m$, where $\gamma = (1 - v^2/c^2)^{-1/2}$ is the Lorentz factor in observer's frame, and the $\gamma$ factor can be absorbed into $\Omega$. Its absolute value is called the gyro-frequency, or Larmor frequency, denoted by $\Omega_L$. Note that in the relativistic regime, gyro-frequency only depends on particle energy irrespective of rest mass. The non-relativistic version of $\Omega_L$ is also called the cyclotron frequency, denoted by $\Omega_c$. We mostly deal with non-relativistic particles in this course.

Gyro-frequency and gyro-radius are of fundamental importance in plasma physics. For reference, we list below the gyro-frequency for electrons $\Omega_e$ and ions $\Omega_i$ in physical units

$$\Omega_e = \frac{17.6}{\gamma_e}\text{s}^{-1}\left(\frac{B}{\mu\text{G}}\right) \ , \quad \Omega_i = \frac{9.58 \times 10^{-3}}{\gamma_i}\text{s}^{-1}\left(\frac{B}{\mu\text{G}}\right) \ , \tag{4}$$

In astrophysics, particle energy is typically measured in eV. Assuming $v_\parallel = 0$, for non-relativistic particles, Larmor radii for electrons and ions are

$$r_{L,e} \approx 33.7 \text{ km}\sqrt{\frac{E}{\text{eV}}}\left(\frac{B}{\mu\text{G}}\right)^{-1} \ , \quad r_{L,i} \approx 1.45 \times 10^3 \text{ km}\sqrt{\frac{E}{\text{eV}}}\left(\frac{B}{\mu\text{G}}\right)^{-1} \ . \tag{5}$$

For relativistic particles (e.g., cosmic-rays), the gyro-radius is only dependent on energy

$$r_L = 0.223\mathrm{AU}\left(\frac{E}{\mathrm{GeV}}\right)\left(\frac{B}{\mu\mathrm{G}}\right)^{-1} = 1.08\times 10^{-6}\mathrm{pc}\left(\frac{E}{\mathrm{GeV}}\right)\left(\frac{B}{\mu\mathrm{G}}\right)^{-1} . \tag{6}$$

These numbers are useful to be kept in mind.

**Plasma diamagnetism**

Gyro-motion of an individual particle is associated with a microscopic current. The mean current averaged over one gyration period is $I = (qv_\perp)/(2\pi r_L)$, and its circular motion gives a *magnetic moment*

$$\boldsymbol{\mu} = -\frac{I}{c}\cdot(\pi r_L^2)\boldsymbol{b} = -\frac{mv_\perp^2}{2B}\boldsymbol{b} = -\mu\boldsymbol{b} . \tag{7}$$

Note that $\boldsymbol{\mu}$ is a vector anti-aligned with the magnetic field (where $\boldsymbol{b}$ is the unit vector along $\boldsymbol{B}$), and is independent of sign of particle charge. It is anti-aligned because ion (electron) gyration is left (right) handed. In this sense, gyration of charged particle around magnetic field is diamagnetic: it tends to produce a magnetic field that reduces the background field.

With a large number of particles, their gyro-motion as a whole gives a magnetization

$$\boldsymbol{M} = n\overline{\boldsymbol{\mu}} = -n\frac{W_\perp}{B^2}\boldsymbol{B} , \tag{8}$$

where $n$ is particle number density, $W_\perp \equiv \overline{mv_\perp^2}/2$ denotes the mean perpendicular kinetic energy of the particles. Note that $\boldsymbol{M}$ has the same dimension as $\boldsymbol{B}$. This magnetization produces a current density

$$\boldsymbol{J}_m = c\nabla\times\boldsymbol{M} , \tag{9}$$

which is called the *magnetization current*. The total current is

$$\boldsymbol{J} = \frac{c}{4\pi}\nabla\times\boldsymbol{B} = \boldsymbol{J}_f + \boldsymbol{J}_m = \boldsymbol{J}_f + c\nabla\times\boldsymbol{M} , \tag{10}$$

where $\boldsymbol{J}_f$ is the current from free charges (due to systematic particle motion) that will be discussed later. Also recall that it is conventional to define the auxiliary field $\boldsymbol{H}\equiv\boldsymbol{B}-4\pi\boldsymbol{M}$, so that $\boldsymbol{J}_f = (c/4\pi)\nabla\times\boldsymbol{H}$.

**Uniform and static B, E field**

Further introducing a constant electric field $\boldsymbol{E}$, the equation of motion (1) can be rearranged into

$$\frac{d\boldsymbol{v}'}{dt} = \frac{q}{m}\left(\boldsymbol{E}_\parallel + \frac{\boldsymbol{v}'}{c}\times\boldsymbol{B}\right) , \tag{11}$$

where

$$\boldsymbol{v}' \equiv \boldsymbol{v} - \boldsymbol{v}_E , \quad \frac{\boldsymbol{v}_E}{c} \equiv \frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2} . \tag{12}$$

Velocity $\boldsymbol{v}_E$ is known as the **$\boldsymbol{E}\times\boldsymbol{B}$ drift velocity**. Particles follow a gyro-motion superposed to the motion of the guiding center, and the latter consists of a constant drift at $\boldsymbol{v}_E$ due to perpendicular electric field $E_\perp$, and a constant acceleration along magnetic field due to parallel electric field $E_\parallel$.

Two points are worth mentioning about the $E\times B$ drift. First, the drift velocity is independent of particle charge, thus it does not produce net current. Second, when $E/B$ reaches or exceeds 1, the notion of $E \times B$ drift fails. In this case, particles are constantly accelerated by $E$ to relativistic velocities, accompanied by increase in their inertia, which was ignored in the derivation above. Unless additional energy loss balances acceleration, particles can never complete a single cycle of gyration.

By the same analogy, particles subject to an external force $\boldsymbol{F}$, such as gravity, will undergo a systematic drift perpendicular $\boldsymbol{F}$ and $\boldsymbol{B}$, at a drift velocity $\boldsymbol{v}_F = (\boldsymbol{F} \times \boldsymbol{B}/qB^2)c$. Note that if this force $\boldsymbol{F}$ is independent of sign of charge, then electrons and ions drift towards opposite directions, creating a current. This is also particularly useful for understanding additional drift effects to be discussed below.

**Inhomogeneous B, E field**

In practice, electromagnetic fields are spatially non-uniform and time varying. If particle gyration time is much shorter than the timescale that fields vary, and particle gyro-radius is much smaller than the spatial scale where fields vary, it is convenient to somewhat neglect the rapid gyration but focus on the motion of the guiding center. The inhomogeneities in background electromagnetic field would introduce small net forces after averaging over particle gyro-phase, leading to slow drift of the guiding center across magnetic field lines.

We decompose particle trajectory into motion of the guiding center, denoted by $\boldsymbol{R}_g$, together with gyro-motion, which is offset from the guiding center by $\boldsymbol{r}_L$. Particle velocity consists of the velocity of the guiding center $\boldsymbol{V}_g$, together with gyro-motion with velocity $\boldsymbol{v}_L$. They satisfy the condition that averaged over gyro-phase, $\langle \boldsymbol{r}_L \rangle = 0$, $\langle \boldsymbol{v}_L \rangle = 0$, and both are perpendicular to magnetic field. The equation of motion now reads

$$\frac{d\boldsymbol{V}_g}{dt} + \frac{d\boldsymbol{v}_L}{dt} = \frac{q}{m}\left[\boldsymbol{E}(\boldsymbol{R}_g + \boldsymbol{r}_L, t) + \frac{(\boldsymbol{V}_g + \boldsymbol{v}_L)}{c} \times \boldsymbol{B}(\boldsymbol{R}_g + \boldsymbol{r}_L, t)\right] . \tag{13}$$

To the zeroth order, averaging over particle gyro-phase, particle guiding center motion follows

$$0 = \boldsymbol{E}(\boldsymbol{R}_g, t) + \frac{\boldsymbol{V}_g^{(0)}}{c} \times \boldsymbol{B}(\boldsymbol{R}_g, t) \quad \Rightarrow \quad \boldsymbol{V}_g^{(0)}(t) = v_\parallel \boldsymbol{b}(t) + \boldsymbol{v}_E(t) . \tag{14}$$

Note that we have assumed that $\boldsymbol{E} \perp \boldsymbol{B}$, which is generally the case in a plasma.

We then proceed to the next order, where we expand the electromagnetic fields to the first order in $\boldsymbol{r}_L$, and $(t - t_0)$, and average over gyro-phase. Note that the time variation in $\boldsymbol{V}_g^{(0)}$ is also considered to be of first order. In the expansion, any term that is linear in $\boldsymbol{r}_L$ and $\boldsymbol{v}_L$ averages to zero, leading to

$$\frac{d\boldsymbol{V}_g^{(0)}}{dt} = \frac{q}{m}\left[\left\langle \frac{\boldsymbol{v}_L}{c} \times (\boldsymbol{r}_L \cdot \nabla \boldsymbol{B})\right\rangle + \frac{\boldsymbol{V}_g^{(1)}}{c} \times \boldsymbol{B}(\boldsymbol{R}_g, t)\right] , \tag{15}$$

where the bracket represents phase-averaging, and we obtain after some algebra

$$\begin{aligned}\langle \boldsymbol{v}_L \times (\boldsymbol{r}_L \cdot \nabla \boldsymbol{B})\rangle &= \Omega \langle (\boldsymbol{r}_L \times \boldsymbol{b}) \times (\boldsymbol{r}_L \cdot \nabla \boldsymbol{B})\rangle \\ &= \Omega \langle \boldsymbol{r}_L \cdot (\boldsymbol{r}_L \cdot \nabla \boldsymbol{B})\boldsymbol{b}\rangle - \langle \boldsymbol{b} \cdot (\boldsymbol{r}_L \cdot \nabla \boldsymbol{B})\boldsymbol{r}_L\rangle \\ &= \Omega \langle \boldsymbol{r}_L \boldsymbol{r}_L\rangle : (\nabla \boldsymbol{B})\boldsymbol{b} - \langle \boldsymbol{r}_L \boldsymbol{r}_L\rangle \cdot \nabla B = -\frac{v_\perp^2}{2\Omega}\nabla B ,\end{aligned} \tag{16}$$

where we have used the fact that

$$\langle \boldsymbol{r}_L \boldsymbol{r}_L \rangle = \frac{v_\perp^2}{2\Omega^2}(\mathsf{I} - \boldsymbol{b}\boldsymbol{b}) , \tag{17}$$

with $\mathsf{I}$ being the identity tensor, and $\mathsf{I} \cdot \nabla \boldsymbol{B} = \nabla \cdot \boldsymbol{B} = 0$. Thus, the first order equation of motion is

$$\frac{\mu}{m}\nabla B + v_\parallel \frac{d\boldsymbol{b}}{dt} + \frac{d\boldsymbol{v}_E}{dt} + \frac{dv_\parallel}{dt}\boldsymbol{b} = \frac{q}{mc}[\boldsymbol{V}_g^{(1)} \times \boldsymbol{B}(\boldsymbol{R}_g, t)] \ , \tag{18}$$

where $\mu \equiv mv_\perp^2/2B$ is magnetic moment.

Let us for now focus on the perpendicular component of Equation (18). Solution to this equation gives the first order drift velocity of the guiding center

$$\boldsymbol{V}_g^{(1)} = \frac{1}{\Omega}\boldsymbol{b} \times \left(\frac{\mu}{m}\nabla B + v_\parallel \frac{d\boldsymbol{b}}{dt} + \frac{d\boldsymbol{v}_E}{dt}\right) \ . \tag{19}$$

The three terms on the right hand side are called *grad B drift*, *inertial drift*, *polarization drift*, respectively. Interpretation of their physical origins is straightforward and discussed below.

<u>Grad $B$ drift</u> occurs in the presence of a gradient in magnetic field strength. When this gradient is perpendicular to $\boldsymbol{B}$, this drift is due to the fact that particle curvature radius varies slightly across one cycle of gyration due to magnetic field gradient. Another way to understand this is that the centripetal force experienced by the particle does not average to zero within one cycle of gyro-motion. The mean force is $\langle \boldsymbol{F} \rangle \propto -\nabla B$, given by the term in the bracket in Equation (15), which leads to a drift along the direction of $-\nabla B \times \boldsymbol{B}$.

For inertial drift, we can first rewrite and expand $d\boldsymbol{b}/dt$ using zeroth order velocity

$$\frac{d\boldsymbol{b}}{dt} = \frac{\partial \boldsymbol{b}}{\partial t} + [(v_\parallel \boldsymbol{b} + \boldsymbol{v}_E) \cdot \nabla]\boldsymbol{b} = \left[\frac{\partial \boldsymbol{b}}{\partial t} + (\boldsymbol{v}_E \cdot \nabla)\boldsymbol{b}\right] + (v_\parallel \boldsymbol{b} \cdot \nabla)\boldsymbol{b} \ . \tag{20}$$

Usually, the last term dominates, and the resulting drift motion is called <u>curvature drift</u>. It occurs when magnetic field is not straight, hence particle guiding center follows the curvy geometry of the field. In particular, note that

$$(\boldsymbol{b} \cdot \nabla)\boldsymbol{b} = \kappa = 1/R_c \ . \tag{21}$$

This motion must be driven by a centripetal force $\boldsymbol{F} \propto d\boldsymbol{b}/dt$, which is in fact provided by the Lorentz force due to curvature drift itself $\boldsymbol{F} = q\boldsymbol{v}_c \times \boldsymbol{B}/c$, where $\boldsymbol{v}_c$ denotes curvature drift velocity. Equivalently, in the frame of guiding center motion, particle is subject to a centrifugal force $\boldsymbol{F} \propto -d\boldsymbol{b}/dt$. The curvature drift is along the direction of $\boldsymbol{F} \times \boldsymbol{B}$.

<u>Polarization drift</u> can be understood as due to a time-varying electric field perpendicular to magnetic field. Suppose sush an electric field is instantaneously applied, it first accelerates electrons and ions towards opposite directions, before both species are picked up by a common $E \times B$ drift velocity. Because ions have much larger Larmor radius, displacement of the ions greatly exceeds that of the electrons, creating an initial polarization of the plasma medium. If the electric field varies continuously in time, the result is a slow drift of electrons and ions in opposite directions that yields a constant change of polarization in the plasma. This current is particularly relevant in plasma waves.

It is worth mentioning that for the three forms of particle drift discussed above, positively and negatively charged particles drift to opposite directions, producing a current. Moreover, the guiding center of particles migrate across field lines, leading to diffusion.

**<u>Plasma current and perpendicular force balance</u>**

Plasma current has three constituents. A streaming current along the magnetic field, a magnetization current discussed earlier, and a current due to drifts. The latter two are perpendicular to magnetic fields. Let us focus on this perpendicular component and discuss the role they play in plasma force balance.

For simplicity, we consider a situation where the plasma is in equilibrium, with all field lines being straight along $\hat{z}$, and has a gradient along $\hat{y}$. The plasma is also uniform in $\hat{z}$. Such a field configuration has a current density

$$\boldsymbol{J}_{\rm tot} = \frac{c}{4\pi}\nabla\times\boldsymbol{B} = \frac{c}{4\pi}\frac{\partial B}{\partial y}\hat{x}\ . \tag{22}$$

This current must be supplied by the plasma. One source of current is from particle drift. The only drift motion in this field configuration is the grad-B drift, which yields a current

$$\boldsymbol{J}_f = \frac{nq}{\Omega}\boldsymbol{b}\times\left(\frac{W_\perp}{mB}\nabla B\right) = -\frac{nW_\perp}{B^2}c\frac{\partial B}{\partial y}\hat{x}\ . \tag{23}$$

In addition, there is the magnetization current

$$\boldsymbol{J}_m = -c\nabla\times\left(n\frac{W_\perp}{B}\boldsymbol{b}\right) = -c\frac{\partial}{\partial y}\left(\frac{nW_\perp}{B}\right)\hat{x}\ . \tag{24}$$

Their sum must equal to $\boldsymbol{J}_{\rm tot}$, and we obtain

$$\boldsymbol{J}_{\rm tot} = \frac{c}{4\pi}\frac{\partial B}{\partial y}\hat{x} = \boldsymbol{J}_f + \boldsymbol{J}_m = -\frac{c}{B}\frac{\partial(nW_\perp)}{\partial y}\hat{x}\ . \tag{25}$$

Note that $nW_\perp$ is just perpendicular pressure $P_\perp$ of the plasma. This equation can be rewritten as

$$\frac{\partial}{\partial y}\left(P_\perp + \frac{B^2}{8\pi}\right) = 0\ . \tag{26}$$

This means that variations in magnetic pressure must be compensated by perpendicular pressure from the plasma. Overall, from microscopic particle gyro-motion, we have arrived at a macroscopic equation of force balance. This will be elaborated further when we discuss magnetohydrodynamics.

**Magnetic mirrors**

Let us consider the component parallel to magnetic field in Equation (18), and simply obtain

$$\frac{d}{dt}(mv_\parallel) = -\mu(\boldsymbol{b}\cdot\nabla B)\ . \tag{27}$$

The right hand side is non-zero when magnetic flux converges or diverges along the field lines. This is called the *magnetic mirror force*, for reasons to become clear shortly.

In the absence of parallel electric field, the Lorentz force does not do work on the particles, and hence particle kinetic energy is conserved. Thus $v_\parallel^2 + v_\perp^2$ is constant

$$\begin{aligned} 0 = \frac{d}{dt}\left(\frac{v_\perp^2}{2}+\frac{v_\parallel^2}{2}\right) &= \frac{d}{dt}\left(\frac{v_\perp^2}{2}\right) - \frac{v_\perp^2}{2B}(v_\parallel\boldsymbol{b}\cdot\nabla B) \\ &= \frac{d}{dt}\left(\frac{v_\perp^2}{2}\right) - \frac{v_\perp^2}{2}\frac{\dot{B}}{B} = \frac{d}{dt}\left(\frac{v_\perp^2}{2B}\right) = \frac{1}{m}\frac{d\mu}{dt}\ . \end{aligned} \tag{28}$$

Therefore, magnetic moment $\mu$ is conserved. The same derivation is applicable to relativistic particles as well (since $\gamma = \text{const}$).

Although our derivation above has assumed no parallel acceleration and no time dependence, conservation of magnetic moment in fact still holds in such general cases, as long as magnetic fields and/or external forces vary much slower than particle gyro-time. Therefore, $\mu$ is called an adiabatic invariant.

Magnetic moment conservation leads to the concept of magnetic mirrors: for a magnetic field configuration where field lines converge towards a narrow area, particles moving towards the converging region tend to get reflected. This is exactly a consequence of energy conservation and $\mu$ conservation. A classic construction of magnetic mirrors is by placing two Helmholtz coils, which generates a "magnetic bottle" configuration.

Let $B_0$ and $B_{\max}$ be the minimum and maximum field strength in the magnetic bottle, which correspond to field strength at the center and bottleneck, and $\theta_0$ be the particle pitch angle at the senter. At other positions with field strength $B$, we have the relation from energy and $\mu$ conservation

$$E_k = \frac{\mu B_0}{\sin^2\theta_0} = \frac{\mu B}{\sin^2\theta} \ . \tag{29}$$

Since $\sin\theta \leq 1$, the maximum field strength a particle can reach is simply $B_1 = B_0/\sin^2\theta_0$. If $B_{\max} > B_1$, then the particle must be reflected before reaching the bottleneck.

**Adiabatic invariants**

When there is a periodicity in particle motion (e.g., gyro-motion), there will be an adiabatic invariant associated with it, as long as the particle trajectory varies very *slowly* compared with the period of this motion. It is an approximation to the more fundamental *Poincaré Invariant* in Hamiltonian mechanics in the presence of periodicities, and takes the form

$$I = \oint_C \boldsymbol{p}\cdot d\boldsymbol{q} \ , \tag{30}$$

where $\boldsymbol{p}$ and $\boldsymbol{q}$ are conjugate canonical variables and the integration is performed along $C$ in phase space over the motion period.

In the case of particle gyro-motion, and using $\phi$ to denote the gyro-phase, we have

$$I = \int_{\phi=0}^{2\pi} \boldsymbol{p}\cdot\frac{d\boldsymbol{q}}{d\phi}d\phi \ . \tag{31}$$

The canonical momentum for charged particles is $\boldsymbol{p} = m\boldsymbol{v} + q\boldsymbol{A}/c$, where $\boldsymbol{A}$ is the vector potential. For gyro-motion,

$$d\boldsymbol{r} = \frac{\partial \boldsymbol{r}_L}{\partial\phi}d\phi = \frac{\boldsymbol{v}_L}{\Omega_L}d\phi \ , \qquad \boldsymbol{A}(\boldsymbol{r}) \approx \boldsymbol{A}(\boldsymbol{R}_g) + (\boldsymbol{r}_L\cdot\nabla)\boldsymbol{A}(\boldsymbol{R}_g) \ . \tag{32}$$

Thus, the action $I$ can be written as

$$\begin{aligned} I &= \int_{\phi=0}^{2\pi} \frac{\boldsymbol{v}_L}{\Omega_L}\cdot\Big\{m(\boldsymbol{V}_g + \boldsymbol{v}_L) + q[\boldsymbol{A} + (\boldsymbol{r}_L\cdot\nabla)\boldsymbol{A}]/c\Big\}d\phi \\ &= 2\pi m\frac{v_\perp^2}{\Omega_L} + 2\pi\frac{q}{c\Omega_L}\langle\boldsymbol{v}_L\cdot(\boldsymbol{r}_L\cdot\nabla)\boldsymbol{A}\rangle \ , \end{aligned} \tag{33}$$

where angle bracket represents averaging over gyro-phase. The latter term can be rewritten as

$$\frac{2\pi q}{c}\langle(\boldsymbol{r}_L\times\boldsymbol{b})\cdot(\boldsymbol{r}_L\cdot\nabla)\boldsymbol{A}\rangle = \frac{2\pi q}{c}\boldsymbol{b}\cdot\langle(\boldsymbol{r}_L\cdot\nabla)\boldsymbol{A}\times\boldsymbol{r}_L\rangle = -\frac{2\pi q}{c}\boldsymbol{b}\cdot\frac{r_L^2}{2}\nabla\times\boldsymbol{A} = -\pi m\frac{v_\perp^2}{\Omega_L} \ . \tag{34}$$

Thus, we have in the end

$$I = 2\pi c \frac{m}{q} \mu \ . \tag{35}$$

Namely, the first adiabatic invariant corresponds to the conservation of magnetic moment.

We have seen that an adiabatic invariant is associated with periodic motions. In magnetic mirror traps, periodic bouncing motion between mirror reflection points introduces a second adiabatic invariant. In this case, we choose the curve $C$ to be the trajectory of the guiding center along bouncing orbit. Since the trajectory is along the magnetic field, we simply have

$$J = \oint p_\parallel ds \ , \tag{36}$$

where the path of integration is along a field line, from the equator to the upper mirror reflection point, back along the field line to the lower reflection point, and then returning to the equator, $ds$ is the arc-length along the field line, $p_\parallel = \boldsymbol{p} \cdot \boldsymbol{b} = m v_\parallel + q A_\parallel / c$. The above expression yields

$$J = m \oint v_\parallel ds + \frac{q}{c} \oint A_\parallel ds = m \oint v_\parallel ds + \frac{q}{c} \Phi_B \ , \tag{37}$$

where $\Phi_B$ is the total magnetic flux enclosed by the bouncing trajectory, which is essentially zero (to a very good approximation because bouncing motion is much faster than drift in the equatorial plane). Thus, the second adiabatic invariant, also known as the longitudinal adiabatic invariant takes the form

$$J = m \oint v_\parallel ds \ . \tag{38}$$

It should be emphasized that this invariant is only a good approximation if magnetic field varies on time-scales much longer than the bouncing time $\tau_b$, which is well satisfied in the Van Allen belts.

In the Earth's Van Allen belts, particles bouncing between the mirror reflection points near the poles also undergo curvature and grad B drifts (see problem set). Conservation of $J$ guarantees that particle must return to the same field line following one cycle of procession. Correspondingly, there is a third adiabatic invariant associated with this periodic motion. We can define a *bounce center* associated with a particle's bouncing motion between mirror reflection points, which lies in the Earth's equatorial plane. We can write the third adiabatic invariant as

$$K \simeq \oint p_\phi ds \ , \tag{39}$$

where the path of integration is the trajectory of the bounce center around the Earth, with $p_\phi = m v_\phi + q A_\phi / c$. The integral is dominated by the second term due to the smallness of the drift velocity $v_\phi$, which becomes

$$K \simeq \frac{q}{c} \oint A_\phi ds = \frac{q}{c} \Phi_B \ , \tag{40}$$

where $\Phi_B$ is the total magnetic flux enclosed by the orbit of bounce center motion around the Earth. Again, $\Phi_B$ is conserved if the Earth's magnetic field varies more slowly than the drift period $\tau_d$, which is on the order of an hour for MeV particles in Van Allen belts. This is only likely the case when the magnetosphere is relatively quiescent.