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Xuening Bai

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Accretion Disks

Accretion disks have been constant main themes of astrophysical research, and many aspects of accretion disks are covered in other astrophysics courses. One may consult an early review by J. E. Pringle (1981, ARA&A, 19, 137), and more recent book *Accretion Power in Astrophysics* by J. Frank, A. King and D. Raine (2002, Cambridge Univ. Press) for most comprehensive reviews of accretion disk physics and observational phenomenologies. Here we only focus on two fundamental MHD processes that play a crucial role in disk angular momentum transport, namely, the magnetorotational instability (MRI) and MHD disk winds.

Angular momentum transport in accretion disks

The set of conservative MHD equations in Cartesian coordinates describe the conservation of mass, momentum and energy. For accretion disk problems, it is the most natural to work in cylindrical coordinates, where the ϕ-component of the momentum equation essentially expresses the conservation of angular momentum. Recall the general form of momentum equation in conservative form

$$\partial_t(\rho\boldsymbol{v}) + \nabla\cdot\mathsf{M} = -\rho\nabla\Phi \tag{1}$$

where Φ is the external gravitational potential (from a point mass in the accretion disk problem), and M is the momentum flux tensor

$$\mathsf{M} \equiv \rho\boldsymbol{v}\boldsymbol{v} - \frac{\boldsymbol{B}\boldsymbol{B}}{4\pi} + \left(P + \frac{B^2}{8\pi}\right)\mathsf{I}\ . \tag{2}$$

In cylindrical coordinates, the momentum equations become

$$\begin{aligned}
\partial_t(\rho v_R) + \frac{1}{R}\partial_R(R\mathsf{M}_{RR}) + \frac{1}{R}\partial_\phi\mathsf{M}_{\phi R} + \partial_z\mathsf{M}_{zR} &= \frac{1}{R}\mathsf{M}_{\phi\phi} - \rho\partial_R\Phi\ , &(3)\\
\partial_t(\rho v_\phi) + \frac{1}{R^2}\partial_R(R^2\mathsf{M}_{R\phi}) + \frac{1}{R}\partial_\phi\mathsf{M}_{\phi\phi} + \partial_z\mathsf{M}_{z\phi} &= -\frac{\rho}{R}\partial_\phi\Phi\ , &(4)\\
\partial_t(\rho v_z) + \frac{1}{R}\partial_R(R\mathsf{M}_{Rz}) + \frac{1}{R}\partial_\phi\mathsf{M}_{\phi z} + \partial_z\mathsf{M}_{zz} &= -\rho\partial_z\Phi\ . &(5)
\end{aligned}$$

Note that in the first equation, the centrifugal force v_ϕ^2/R is reflected in the $M_{\phi\phi}$ term which balances gravity from the central object. The ϕ-momentum equation can further be reduced to

$$\partial_t(\rho R v_\phi) + \frac{1}{R}\partial_R(R^2\mathsf{M}_{R\phi}) + \frac{1}{R}\partial_\phi(R\mathsf{M}_{\phi\phi}) + \partial_z(R\mathsf{M}_{z\phi}) = -\rho\partial_\phi\Phi\ , \tag{6}$$

which essentially expresses angular momentum conservation.

Now let us assume the disk is thin and integrate this equation over height and azimuth. Note that the $\partial\phi$ terms vanish after integration over ϕ, and the diagonal component of M is irrelevant to the angular momentum fluxes. The result is

$$\frac{\partial(2\pi R\Sigma j_0)}{\partial t} + \frac{\partial}{\partial R}\left[2\pi R^2\int_{-\infty}^{\infty}dz\left(\overline{\rho v_R v_\phi} - \frac{\overline{B_R B_\phi}}{4\pi}\right)\right] + 2\pi R^2\left(\overline{\rho v_z v_\phi} - \frac{\overline{B_z B_\phi}}{4\pi}\right)\Bigg|_{-\infty}^{\infty} = 0\ , \tag{7}$$

where $j_0 \approx v_K R$ is the specific angular momentum at radius R, $\Sigma = \int \rho dz$ is the disk surface density, and the over line denotes the azimuthal average. The radial and azimuthal velocities can be further decomposed into $v_R = v_{R0} + \delta v_R$, $v_\phi = v_{\phi 0} + \delta v_\phi$, where subscript "$_0$" represents the mean, and δ represents the fluctuations.

$$\frac{\partial(2\pi R\Sigma j_0)}{\partial t}+\frac{\partial(\dot{M}_{\rm acc} j_0)}{\partial R}+\frac{\partial}{\partial R}\left[2\pi R^2\int_{-\infty}^{\infty}dz\left(\overline{\rho\delta v_R\delta v_\phi}-\frac{\overline{B_RB_\phi}}{4\pi}\right)\right]+2\pi R^2\left(\overline{\rho v_z v_\phi}-\frac{\overline{B_zB_\phi}}{4\pi}\right)\bigg|_{-\infty}^{\infty}=0\ , \quad (8)$$

where $\dot{M}_{\rm acc} = 2\pi R\int \rho v_{R0}dz$ is the accretion rate. Note that accretion corresponds to $\dot{M}_{\rm acc} < 0$.

The terms in the first bracket denote angular momentum flux in the radial direction. They are called the Reynolds stress and Maxwell stress, respectively. Whether this flux leads to accretion is determined by the sign of its radial gradient. Driving accretion requires outward transport of angular momentum, leading to the accretion of most of the disk mass, with a small amount of the outer disk mass receiving angular momentum and spread out. Radial transport of angular momentum is also called "viscous transport", because a viscous fluid can supply the Reynolds stress from Keplerian shear. However, in astrophysical disks, microscopic viscosity is too small to be relevant, and generating such stresses generally requires turbulence. There has been a long debate whether a pure hydrodynamic mechanism can become unstable to produce turbulence, but most recent laboratory experiments have demonstrated that Keplerian rotation is stable even at very high Reynolds numbers, consistent with Rayleigh criterion. Since the (re-)discovery of the magneto-rotational instability (MRI) by Balbus & Hawley (1991), the community has quickly arrived at the consensus that MRI is the most powerful mechanism of driving turbulence in most accretion disks and to transport angular momentum. It is common to parameterize the stress using the α prescription (Shakura & Sunyaev, 1973), writing

$$\overline{\rho\delta v_R\delta v_\phi}-\frac{\overline{B_RB_\phi}}{4\pi}\equiv\alpha P\approx\alpha\rho c_s^2\ , \quad (9)$$

where c_s is the isothermal sound speed. The dimensionless α parameter provides a very useful for the efficiency of angular momentum transport. Typical values of α inferred from observations is of the order 0.1 (King et al., 2007).

The terms in the second bracket denote angular momentum flux in the vertical direction through disk surface, carried by a disk wind. Note that the $\rho v_z v_\phi$ term simply represents the loss of angular momentum possessed by the outflowing materials (since $v_\phi \sim v_K$). In other words, no additional angular momentum is extracted from the disk. On the other hand, the B_zB_ϕ term represents the *extraction* of *excess* angular momentum from materials still residing in the disk. This is the essence of angular momentum transport by an MHD wind.

Magneto-rotational Instability

A very concise and physically intuitive description of the MRI physics with can be found in the scholarpedia article by Steven A. Balbus. Here we provide a more formal derivation using *Eulerian* approach.

We consider the local shearing-sheet approximation (Goldreich & Lynden-Bell, 1965) of the MHD equations. In this approximation, we take a local patch of a disk centered on radius R_0, and work in a frame corotating with the disk at this radius, which has angular velocity Ω_K. The advantage of this approach is that we can ignore disk curvature, and use the Cartesian coordinate system. By convention, x, y, and z correspond to radial, azimuthal and vertical dimensions, with $x = 0$ at $R = R_0$, and $\boldsymbol{\Omega}_K$ in the $\boldsymbol{e}_z$ direction. Velocity $\boldsymbol{v}$ in this frame is the velocity relative to rotation velocity at R_0. In this corotating frame, two non-inertial forces must be included, namely, the Coriolis force $2\boldsymbol{v} \times \boldsymbol{\Omega}_K$ and the centrifugal force $\Omega_K^2(R_0 + x)\boldsymbol{e}_x$. In addition, we expand the gravitational force around this radius (here we ignore the vertical component of gravity)

$$\boldsymbol{F}_g(x) = -\frac{GM}{(R_0+x)^2}\boldsymbol{e}_x \approx -\frac{GM}{R_0^2}\left(1 - 2\frac{x}{R_0}\right)\boldsymbol{e}_x = (-\Omega_K^2 R_0 + 2\Omega_K^2 x)\boldsymbol{e}_x \ . \tag{10}$$

Having all the forces combined, MHD equations in a shearing-sheet read

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}) = 0 \ , \\ \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\frac{\nabla P}{\rho} + [2\boldsymbol{v}\times\boldsymbol{\Omega}_K + 3\Omega_K^2 x\boldsymbol{e}_x] + \frac{1}{4\pi\rho}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} \ , \\ \frac{\partial \boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{v}\times\boldsymbol{B}) \ . \end{aligned} \tag{11}$$

The source terms on the right hand side of the momentum equation correspond to Coriolis force and tidal force, respectively. We have ignored the energy equation since it is irrelevant to the MRI physics. In the background state, let everything be uniform with $\rho = \rho_0$, $P = P_0$. There is a background vertical magnetic field $\boldsymbol{B} = B_0\boldsymbol{e}_z$. The background velocity follows from the balance between the Corilois and tidal forces, giving Keplerian rotation

$$\boldsymbol{v}_0 = -\frac{3}{2}\Omega x\boldsymbol{e}_y \ . \tag{12}$$

The physics of the MRI does not involve compressible modes. We thus consider only *incompressible* perturbations, namely, $\rho = \rho_0$ being constant. We write $\boldsymbol{v} = \boldsymbol{v}_0 + \delta\boldsymbol{v}$, and $\boldsymbol{B} = B_0\boldsymbol{e}_z + \delta\boldsymbol{B}$. To the first order, the perturbation equations read

$$\begin{aligned} &\nabla\cdot(\delta\boldsymbol{v}) = 0 \ , \\ &\frac{\partial\delta\boldsymbol{v}}{\partial t} = -\frac{\nabla\delta P}{\rho_0} - \frac{1}{2}\Omega_K\delta v_x\boldsymbol{e}_y + 2\Omega_K\delta v_y\boldsymbol{e}_x + \frac{1}{4\pi\rho_0}(\nabla\times\delta\boldsymbol{B})\times\boldsymbol{B}_0 \ , \\ &\frac{\partial\delta\boldsymbol{B}}{\partial t} = \nabla\times(\delta\boldsymbol{v}\times\boldsymbol{B}_0) - \frac{3}{2}\Omega_K\delta B_x\boldsymbol{e}_y \ . \end{aligned} \tag{13}$$

We assume perturbations are axisymmetric in the form of $\exp(ikz + \sigma t)$. Furthermore, we nondimensionalize the variables as follows

$$\tilde{\sigma} \equiv \sigma/\Omega_K \ , \quad \tilde{\boldsymbol{v}} \equiv \boldsymbol{v}/v_A \ , \quad \boldsymbol{b} \equiv \boldsymbol{B}/B_0 \ , \quad \tilde{k} \equiv k v_A/\Omega_K \ , \tag{14}$$

where the background Alfvén velocity $v_A = B_0/\sqrt{4\pi\rho_0}$. Without loss of clarity, however, we omit the $\tilde{}$ sign in the derivation that follows.

The continuity equation gives

$$k\delta v_z = 0 \ , \quad \text{or} \quad \delta v_z = 0 \ . \tag{15}$$

In fact, this relation is decoupled from the rest of the equations and is unused.

For the momentum equation, the pressure perturbation is along the direction $\boldsymbol{k}$, and it suffices to consider perturbations perpendicular to $\boldsymbol{k}$ (to get rid of the pressure term):

$$\sigma(\boldsymbol{k}\times\delta\boldsymbol{v})-2\delta v_y k\boldsymbol{e}_y-\delta v_x k\boldsymbol{e}_x/2=\mathrm{i}k(\boldsymbol{k}\times\delta\boldsymbol{b})\ , \tag{16}$$

which can be rewritten as

$$\begin{pmatrix}\sigma & -2\\ 1/2 & \sigma\end{pmatrix}\begin{pmatrix}\delta v_x\\ \delta v_y\end{pmatrix}=\mathrm{i}k\begin{pmatrix}\delta b_x\\ \delta b_y\end{pmatrix}\ . \tag{17}$$

Note that in the hydrodynamic limit, no magnetic field is involved and hence $\delta b_x=\delta b_y=0$. The above equations simply describe *epicyclic oscillation*, requiring $\sigma^2=-1$, i.e., there is no instability. The oscillation frequency is $\kappa=\Omega_K$ for Keplerian rotation.

For the perturbation to the induction equation, we obtain

$$\sigma\delta\boldsymbol{b}+(3/2)\delta b_r\boldsymbol{e}_y=\mathrm{i}\boldsymbol{k}\times(\delta\boldsymbol{v}\times\boldsymbol{e}_z)\ . \tag{18}$$

which can be rewritten as

$$\begin{pmatrix}\sigma & 0\\ 3/2 & \sigma\end{pmatrix}\begin{pmatrix}\delta b_x\\ \delta b_y\end{pmatrix}=\mathrm{i}k\begin{pmatrix}\delta v_x\\ \delta v_y\end{pmatrix}\ . \tag{19}$$

The dispersion relation can be obtained by combining equations (17) and (19), and requiring the determinant of the resulting matrix to vanish. We can obtain

$$\sigma^4+\sigma^2(2k^2+1)+k^2(k^2-3)=0\ . \tag{20}$$

This relation has real and positive roots (unstable) when $0<k^2<3$, given by

$$\sigma^2=\frac{-(2k^2+1)+\sqrt{16k^2+1}}{2}\ . \tag{21}$$

By taking derivative on k, the fastest growth rate is achieved at $k^2=15/16$, with $\sigma=3/4$. Restoring physical units, the range of unstable wavenumber is $0<k<k_c$, where $k_c=\sqrt{3}\Omega_K/v_A$ is called the *critical wavenumber*. The *most unstable wavenumber* is $k_m=\sqrt{15/16}\Omega_K/v_A$, at which the growth rate reaches maximum value of $\sigma^{\max}=(3/4)\Omega_K$.

In the derivation above, we have restricted ourselves to Keplerian disks with $\Omega(R)\propto R^{-3/2}$. For more general rotation profiles, the dispersion relation reads

$$\sigma^4+\sigma^2[\kappa^2+2(kv_A)^2]+(kv_A)^2\left[(kv_A)^2+\frac{d\Omega^2}{d\ln R}\right]=0\ , \tag{22}$$

where $\kappa^2\equiv4\Omega^2+d\Omega^2/d\ln R=d(R^2\Omega)^2/d\ln R$ is epicyclic frequency squared, with $\kappa=\Omega_K$ for Keplerian rotation. Thus, the criterion for instability is given by

$$\frac{d\Omega}{dR}<0\ . \tag{23}$$

This is in stark contrast with the *Rayleigh criterion* in the hydrodynamic limit, where shear-flow becomes unstable when $d(R^2\Omega)/dR<0$ (i.e., epicyclic frequency becomes imaginary), which is hopeless for accretion disks.

It is worth noting that the stability criterion (23), and the fastest growth rate $\sigma^{\max}$ has no dependence on magnetic field strength. Therefore, introduction of a vertical magnetic field, no matter how weak it is, would completely change the stability criterion of the flow and yield exponential growth with e-folding time being almost the dynamical time! This apparent dilemma is resolved by noting that as $B_0 \to 0$, the most unstable wavelength shifts towards $k \to \infty$. At certain point before B_0 reaching zero, the dispersion relation above would fail because ideal MHD is no longer applicable at very small scales (e.g., resistive dissipation, or kinetic effects).

MHD Disk Winds

MHD disk winds fall into a broad subject of MHD winds and jets produced by rotating objects with magnetic fields anchored on them. Other examples include stellar winds, pulsar winds, jets from gamma-ray burst, etc. The article by H. C. Spruit (1996, astro-ph 9602022) provides a very pedagogical review of the basic physics of MHD winds. Here we simply introduce the fundamental elements of the MHD disk wind theory.

The general properties of MHD winds can be obtained from considering steady-state, axisymmetric MHD equations. We can write the magnetic and velocity fields

$$\boldsymbol{B} = \boldsymbol{B}_p + B_\phi \boldsymbol{e}_\phi \ , \qquad \boldsymbol{v} = \boldsymbol{v}_p + R\Omega(R)\boldsymbol{e}_\phi \ , \tag{24}$$

where subscprit p denotes poloidal component, $\boldsymbol{e}_\phi$ is a unit vector along the toroidal direction, R is cylindrical radius, and Ω is angular velocity of the flow.

By axisymmetry and $\nabla \cdot \boldsymbol{B}_p = 0$, poloidal field can be expressed in terms of a *flux function*

$$\boldsymbol{B}_p = \frac{1}{R}\nabla\psi \times \boldsymbol{e}_\phi \ . \tag{25}$$

Obviously, the flux function is constant along field lines

$$\boldsymbol{B}_p \cdot \nabla\psi = 0 \ . \tag{26}$$

Thus, ψ can be considered as a label of individual field lines.

In ideal MHD, the induction equation is given by

$$\nabla \times [(\boldsymbol{v}_p + v_\phi \boldsymbol{e}_\phi) \times (\boldsymbol{B}_p + B_\phi \boldsymbol{e}_\phi)] = 0 \ . \tag{27}$$

Equivalently, we have

$$(\boldsymbol{v}_p \times \boldsymbol{B}_p) + (v_\phi \boldsymbol{e}_\phi \times \boldsymbol{B}_p + B_\phi \boldsymbol{v}_p \times \boldsymbol{e}_\phi) = \nabla f \ , \tag{28}$$

where f is an axisymmetric scalar function.

By axisymmetry, the toroidal component in (28) must be zero, hence

$$\boldsymbol{v}_p \times \boldsymbol{B}_p = 0 \ , \quad \to \qquad \boldsymbol{v}_p = \alpha \boldsymbol{B}_p \ , \tag{29}$$

where α is a scalar. Using this result, consider the dot product of $\boldsymbol{B}_p$ with (28), we find $\boldsymbol{B}_p \cdot \nabla f = 0$. This means that f is constant along poloidal field lines, hence we can think of f as a function of ψ.

Taking the cross product of $\boldsymbol{e}_\phi$ with (28), and using $\boldsymbol{v}_p = \alpha \boldsymbol{B}_p$, we obtain

$$\Omega - \frac{\alpha B_\phi}{R} = \frac{df}{d\psi} \equiv \omega \ , \tag{30}$$

where ω is constant along a particular field line. Note that when $\alpha = 0$ (no mass loading), equation (30) reduces to Ferraro's law of isorotation $\Omega = \omega$. Physically, one expects $B_\phi = 0$ at the disk midplane by symmetry. Therefore, ω is the angular velocity of the field line at the disk midplane.

From the continuity equation, we have

$$\nabla \cdot (\rho \boldsymbol{v}) = \nabla \cdot (\rho \alpha \boldsymbol{B}) = \boldsymbol{B} \cdot \nabla \frac{k}{4\pi} = 0 \ , \tag{31}$$

where

$$k \equiv 4\pi\rho\alpha = \frac{4\pi\rho v_p}{B_p} \tag{32}$$

being constant along a field line. The physical meaning of k is obvious: it gives the ratio of mass flux to magnetic flux.

With the results above, we see that along a particular field line, the flow velocity can be conveniently expressed as

$$\boldsymbol{v} = \frac{k\boldsymbol{B}}{4\pi\rho} + \omega R \boldsymbol{e}_\phi \ . \tag{33}$$

Therefore, in the frame corotating with the foot point of a field line, the flow is everywhere parallel to the magnetic field.

The toroidal component of the momentum equation can be written as (in cylindrical coordinate)

$$R\left[\frac{(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}}{4\pi}\right]_t = \rho \boldsymbol{v} \cdot \nabla(\Omega R^2) \ . \tag{34}$$

With the help of equation (33), one finds

$$\boldsymbol{B} \cdot \nabla \left[R B_\phi - k \Omega R^2 \right] = 0 \ , \tag{35}$$

Therefore, we can define

$$l \equiv \Omega R^2 - \frac{R B_\phi}{k} = \Omega R^2 - \frac{R B_\phi B_p}{4\pi\rho v_p} \ . \tag{36}$$

The physical meaning of l is simply the specific angular momentum in the wind flow, which is conserved along a field line. In particular, the factor $B_\phi B_p$ directly corresponds to the $B_\phi B_z$ term in the angular momentum conservation equation (8), expressing angular momentum exchange between matter and magnetic fields via the magnetic torque.

One can substitute B_ϕ from equation (33) to equation (36), and using equation (32) to obtain

$$v_\phi - \omega R = \frac{l - \omega R^2}{R(1 - 4\pi\rho/k^2)} \ . \tag{37}$$

The right hand side of the equation is singular when $4\pi\rho = k^2$, i.e., $v_p^2 = B_p^2/4\pi\rho = v_{Ap}^2$. This corresponds to the Alfvén point, where the poloidal flow velocity equals to the poloidal Alfvén speed v_{Ap}. The cylindrical radius of the Alfvén point, R_A, is called the Alfvén radius. For the flow to pass through the Alfvén point smoothly, the numerator of the above equation must vanish, which yields

$$l = \omega R_A^2 \ . \tag{38}$$

The Alfvén radius characterizes the efficiency of the wind for extracting disk angular momentum. Since in general $\omega \approx \Omega_K$, the excess angular momentum per unit mass in the wind is $\Omega_K(R_A^2 - R_0^2)$ (where R_0 is the radius of the wind footpoint), which is extracted from the disk. Let $\dot{M}_{\rm acc}$ be the wind-driven accretion rate at radius R_0. Angular-momentum conservation leads to

$$\dot{M}_{\rm acc}\frac{dj}{dR} = \frac{d\dot{M}_{\rm wind}}{dR}\Omega_K(R_A^2 - R_0^2) \ , \tag{39}$$

where $j(R) \equiv \Omega_K R^2$ is the specific angular momentum in the disk, and $\dot{M}_{\rm wind}(R)$ denotes the cumulative mass loss rate from the origin to disk radius R. Noting that $d\ln j/d\ln R = 1/2$ in Keplerian disks, we obtain

$$\xi \equiv \frac{d\dot{M}_{\rm wind}/d\ln R}{\dot{M}_{\rm acc}}\bigg|_{R=R_0} = \frac{1}{2}\frac{1}{(R_A/R_0)^2 - 1} \ . \tag{40}$$

This number is sometimes called the "ejection index", and we see that the ratio of wind mass loss rate to wind-driven accretion rate is intimately connected to the Alfvén radius.

Assuming adiabaticity, the energy equation reads

$$\nabla\cdot\left[\rho\boldsymbol{v}\left(\frac{1}{2}v^2 + h + \Phi\right) + \frac{B^2\boldsymbol{v} - (\boldsymbol{B}\cdot\boldsymbol{v})\boldsymbol{B}}{4\pi}\right] = 0 \ , \tag{41}$$

and after some manipulation using (30), we find

$$\boldsymbol{B}\cdot\nabla\left[k\left(\frac{v^2}{2} + h + \Phi\right) - \omega R B_\phi\right] = 0 \ , \tag{42}$$

which defines a fourth conserved quantity

$$e \equiv \frac{v^2}{2} + h + \Phi - \frac{\omega R B_\phi}{k} \ , \tag{43}$$

being the specific energy along a field line. In the above, $h \equiv \int dP/\rho$ is specific enthalpy, Φ is gravitational potential, and the last term represents the work done on the streaming gas by magnetic torque.

The energy equation (43) can be combined with the angular momentum equation (36) to yield the Bernoulli equation

$$E \equiv e - \omega l = \frac{v^2}{2} - \omega R v_\phi + h + \Phi = \frac{v_p^2 + (v_\phi - \omega R)^2}{2} + h + \Phi_{\rm eff} \ , \tag{44}$$

where $\Phi_{\rm eff} \equiv \Phi - \omega^2 R^2/2$ is the effective potential, and E again is a constant along any field line. An important feature about the Bernoulli equation is that E is independent of field variables.

In sum, along a field line, the MHD wind is characterized by four constants of motion as defined in equations (30), (32), (36) and (43). These four constants are not independent of each other. In particular, the axisymmetric MHD wind problem has two *critical points*, which occur when poloidal flow speed matches the slow and fast magnetosonic speeds propagating along the field. The requirement that the flow pass smoothly through each of these points imposes additional constraints that reduces two degrees of freedom. The specific energy is essentially determined by conditions at the wind launching region. See Spruit's review for further details.

The four conservations equations discussed above completely specify wind properties along a prescribed poloidal field line. But what determines the shape of the field lines? We need to solve the *cross-field force balance*: the force balance perpendicular to each field line (or magnetic flux surface ψ =const), which depends on the shape and wind properties along each field line, as well as neighboring field lines. Therefore, solving the structure of an MHD disk wind is intrinsically a global problem. Mathematically, it is described by the *Grad-Shafranov equation*, which is a second-order non-linear partial differential equation on the magnetic flux function ψ. It is of different types (elliptical/hyperbolic) in between different critical surfaces, and is very challenging to solve. Special solutions has been obtained by inserting additional *ansatz* that greatly simplify the equation, most notably the self-similar "magneto-centrifugal" wind solution by Blandford & Payne (1982). More flexible wind solutions can be obtained from MHD simulations. It is known that wind properties mainly depends on 1). strength of poloidal magnetic field threading the disk, and distribution of magnetic flux; 2). thermodynamics near the wind footpoint (launching region). Depending on the relative importance of magnetic field and thermal pressure, the wind can be driven by centrifugal effect (for strong field with inclined poloidal field geometry), or by magnetic pressure gradient of the toroidal field that is built up by twisting the poloidal field (the weak poloidal field case). Because toroidal field becomes more and more dominant towards larger distance, MHD winds can be self-collimated beyond the Alfvén point by "magnetic hoop stress".

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