Vlasov-Maxwell Equations and Cold Plasma Waves

The Vlasov-Maxwell equations

Consider a plasma as a collection of N charged particles, each particle i has position and velocity $\boldsymbol{x}_i$ and $\boldsymbol{v}_i$. By definition,

$$\dot{\boldsymbol{x}}_i = \boldsymbol{v}_i \ , \quad \dot{\boldsymbol{v}}_i = \boldsymbol{a}_i = \frac{q}{m}\left(\boldsymbol{E} + \frac{\boldsymbol{v}_i \times \boldsymbol{B}}{c}\right) \ . \tag{1}$$

Rather than solving this entire array of $6N$ equations, we simplify the problem by adopting a statistical treatment based on the assumption that we do not distinguish between particles (of the same species), if they are at about the same position and share about the same velocity. Hence, we define $f(\boldsymbol{x}, \boldsymbol{v}, t)$ as the particle distribution function, which represents the number density of particles found near the point $(\boldsymbol{x}, \boldsymbol{v})$ in phase space. Specifically, the number of particles located within intervals d^3x about $\boldsymbol{x}$ and d^3v about $\boldsymbol{v}$ is given by

$$dN = f(\boldsymbol{x}, \boldsymbol{v}, t)d^3x d^3v \ . \tag{2}$$

The 6D space whose volume element is $d^3x d^3v$ is called "phase space".

If there were no particle creation or destruction (e.g., ionization/recombination), the total number of charged particles must be conserved. Analogous to how we deal with conservation laws in MHD, conservation of particles in phase space can be expressed as

$$\frac{\partial f(\boldsymbol{X}, t)}{\partial t} + \frac{\partial}{\partial \boldsymbol{X}}(f\dot{\boldsymbol{X}}) = 0 \ , \tag{3}$$

where $\boldsymbol{X} \equiv (\boldsymbol{x}, \boldsymbol{v})$ is phase space coordinates and $\dot{\boldsymbol{X}} = (\boldsymbol{v}, \boldsymbol{a})$ is "velocity" in phase space. More explicitly, that is

$$\frac{\partial f}{\partial t} + \nabla \cdot (f\boldsymbol{v}) + \nabla_{\boldsymbol{v}} \cdot (f\boldsymbol{a}) = 0 \ . \tag{4}$$

Obviously, $\boldsymbol{v}$ has no dependence on $\boldsymbol{x}$, hence $\nabla \cdot (f\boldsymbol{v}) = \boldsymbol{v} \cdot \nabla f$. However, $\boldsymbol{a}$ does depend on $\boldsymbol{v}$ due to the $\boldsymbol{v} \times \boldsymbol{B}$ term. Fortunately, because $\nabla_{\boldsymbol{v}} \cdot (\boldsymbol{v} \times \boldsymbol{B}) = (\nabla_{\boldsymbol{v}} \times \boldsymbol{v}) \cdot \boldsymbol{B} - \boldsymbol{v} \cdot (\nabla_{\boldsymbol{v}} \times \boldsymbol{B}) = 0$, we can also take $\boldsymbol{a}$ out from the the divergence operator. Thus, the conservation equation becomes

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f + \boldsymbol{a} \cdot \nabla_{\boldsymbol{v}} f = 0 \ , \tag{5}$$

or, written in more compact form,

$$\frac{Df}{Dt} \equiv \left(\frac{\partial}{\partial t} + \dot{\boldsymbol{X}}\frac{\partial}{\partial \boldsymbol{X}}\right)f = 0 \ , \tag{6}$$

where D/Dt is the convective derivative which follows particle trajectories in phase space. The result of $Df/Dt = 0$ is known as *Liouville's theorem*. It states that the distribution function f is constant along particle trajectories in phase space (when $\nabla_{\boldsymbol{v}} \cdot \boldsymbol{a} = 0$).

Written in full, we have the kinetic equation describing the evolution of f:

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f + \frac{q}{m}\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}\right) \cdot \nabla_{\boldsymbol{v}} f = 0 \ . \tag{7}$$

The driving force of evolution comes from acceleration $\boldsymbol{a}$, which depends on fields $\boldsymbol{E}$ and $\boldsymbol{B}$. Typically, the electromagnetic fields are partially externally applied, and partially created by plasma charge and current. Contribution from the latter can further be roughly divided into two parts. One is the *collective* response of particles to electromagnetic fields, which is smooth; and the other part is from small-scale inter-particle interactions in nature (i.e., Coulomb collisions within λ_D). In plasma physics, it is convenient to combine the externally applied fields with fields generated by collective plasma response to evaluate the acceleration $\boldsymbol{a}$. In other words, we only consider the smooth part of the $\boldsymbol{E}$, $\boldsymbol{B}$ fields (this is possible because it does not depend on the exact particle location). The remaining part, force due to small-scale fields from near-neighbor particle interactions, is considered as "collisions" and is reflected in an unspecified term on the right side:

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f + \frac{q}{m}\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}\right) \cdot \nabla_{\boldsymbol{v}} f = \left(\frac{\partial f}{\partial t}\right)_{\text{coll}} \ . \tag{8}$$

This is called the *Boltzman Equation*. When this term can be ignored (depending on the plasma parameters and the scales of the problem), we are left with the "collisionless Boltzman equation", more commonly referred to as the *Vlasov equation*.

Evolution of f in the Boltzman equation must be accompanied by the equations to evolve the fields, described by Maxwell's Equations.

$$\frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} = \nabla \times \boldsymbol{B} - \frac{4\pi}{c}\boldsymbol{j} \ , \tag{9a}$$

$$\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E} \ , \tag{9b}$$

$$\nabla \cdot \boldsymbol{E} = 4\pi\rho \ , \tag{9c}$$

$$\nabla \cdot \boldsymbol{B} = 0 \ . \tag{9d}$$

Evolution of electromagnetic fields are coupled to electric charges and current densities ρ and $\boldsymbol{j}$, which are moments of the distribution function

$$\rho = \sum_{\text{species}} q \int f(\boldsymbol{x}, \boldsymbol{v}, t) d^3 v \ ,$$
$$\boldsymbol{j} = \sum_{\text{species}} q \int \boldsymbol{v} f(\boldsymbol{x}, \boldsymbol{v}, t) d^3 v \ . \tag{10}$$

Note that not all of Maxwell's equations are independent. The induction equation (9b) implies divergence free condition of the magnetic field (9d). Equations (9a) and (9c) imply charge conservation, but that is already built-in to the Vlasov equation. Thus, only one of the two is needed. In particular, if no magnetic field is involved, one can conveniently use (9c), and the corresponding modes are called *electrostatic* modes. Otherwise, one generally uses (9a) instead.

The plasma dielectric tensor

Useful insight can be gained even without solving the Vlasov equation. From Maxwell's equations

$$\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E} \ ,$$
$$\frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} = \nabla \times \boldsymbol{B} - \frac{4\pi}{c}\boldsymbol{j} \ . \tag{11}$$

Note that the "displacement current", or the $\partial \boldsymbol{E}/\partial t$ term, is ignored in MHD. Whether we would like to keep it or not depends on the problem of interest. Retaining this term, one would recover electromagnetic (light) waves and plasma oscillations, while removing this term one generally filters out these high-frequency modes, and misses the electron-scale (skin depth d_e) physics.

Suppose the background plasma has a uniform field $\boldsymbol{B}_0$ and no electric field (we can always choose a proper frame to satisfy this). Without loss of generality, we take $\boldsymbol{B}_0$ to be along the $\boldsymbol{e}_z$ axis. Consider perturbations in the general form of $\exp\left[i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)\right]$, and we add subscript "1" to perturbed fields. The above equations turn into

$$-\frac{i\omega}{c}\boldsymbol{B}_1 = -i\boldsymbol{k}\times\boldsymbol{E}_1 \ , \tag{12a}$$

$$-\frac{i\omega}{c}\boldsymbol{E}_1 = i\boldsymbol{k}\times\boldsymbol{B}_1 - \frac{4\pi}{c}\boldsymbol{j}_1 \ . \tag{12b}$$

The current density $\boldsymbol{j}_1$ reflects the response of the plasma to electromagnetic perturbations, and generally requires one to solve the Vlasov equation. The solution to this equation gives a relation between $\boldsymbol{j}_1$ and $\boldsymbol{E}_1$. Once this relation is known, we can define the dielectric tensor ϵ as follows [1]

$$\epsilon\cdot\boldsymbol{E}_1 = \boldsymbol{E}_1 + i\frac{4\pi}{\omega}\boldsymbol{j}_1 \ . \tag{14}$$

In most plasma wave/instability problems, the main task generally involves the calculation of ϵ from the Vlasov equation.

With ϵ at hand, Equation (12b) becomes

$$-\frac{i\omega}{c}\epsilon\cdot\boldsymbol{E}_1 = i\boldsymbol{k}\times\boldsymbol{B}_1 \ . \tag{15}$$

Combining the two Maxwell equations, we obtain

$$\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{E}_1) + \frac{\omega^2}{c^2}\epsilon\cdot\boldsymbol{E}_1 = 0 \ . \tag{16}$$

Equivalently, we can write it in tensor form

$$L_{ij}E_j = 0 \ , \quad \text{where } L_{ij} \equiv k_ik_j - k^2\delta_{ij} + \frac{\omega^2}{c^2}\epsilon_{ij} \ . \tag{17}$$

This equation has non-trivial solutions only when the determinant of the matrix L_{ij} vanishes. This is a polynomial equation for ω as a function of $\boldsymbol{k}$. Note that the dielectric tensor ϵ_{ij} is in general also a function

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[1]Recall that in electrodynamics in continuum medium, the dielectric tensor is defined as

$$\epsilon\cdot\boldsymbol{E} = \boldsymbol{E} + 4\pi\boldsymbol{P} \ , \tag{13}$$

where $\boldsymbol{P}$ is called "polarization", and associated with it is the polarization current $\boldsymbol{j} = \partial\boldsymbol{P}/\partial t$.

of ω and $\boldsymbol{k}$. Each solution of $\omega(\boldsymbol{k})$ of this equation represents the dispersion relation of a particular wave mode. Therefore, we can regard it as the general plasma dispersion relation for plasma waves of any kind. In particular, when taking $\epsilon = 1$ (no plasma current response), it describes the propagation of light waves in vacuum. On the other hand, if no perturbations in magnetic fields are involved, such modes are called *electrostatic* modes, and the dispersion relation is simply given by $\epsilon = 0$.

Waves in cold plasmas

While plasma response should be rigorously derived from the Vlasov equation, the situation is greatly simplified in a "cold" plasma, where particles have essentially no random velocities.[2] Thus, the motions of all electrons/ions in the wave field are identical. This means that treating individual particles species as a (pressureless) fluid and solve for its motion is equivalent to solving the Vlasov equation.

Suppose that in the background state, the plasma is homogeneous with n_{0s} being the number density of species s, and all particles are static $\boldsymbol{v}_0 = 0$. Charge neutrality ensures that $\sum_s n_{0s} q_s = 0$. Perturbed quantities are denoted with subscript "1". Let background field $\boldsymbol{B}_0$ be along the $\hat{z}$ direction. The response from particle species s can be obtained from pressureless fluid equations. The linearized fluid equations for individual particle species are

$$\frac{\partial n_{1s}}{\partial t} + n_{0s} \nabla \cdot \boldsymbol{v}_{1s} = 0 \;,$$
$$\frac{\partial \boldsymbol{v}_{1s}}{\partial t} = \frac{q_s}{m_s} \left(\boldsymbol{E}_1 + \frac{\boldsymbol{v}_{1s}}{c} \times \boldsymbol{B}_0 \right) \;. \tag{18}$$

The perturbations will give a first-order current density

$$\boldsymbol{j}_1 = \sum_s q_s (n_{1s} \boldsymbol{v}_{0s} + n_{0s} \boldsymbol{v}_{1s}) = \sum_s q_s n_{0s} \boldsymbol{v}_{1s} \;. \tag{19}$$

Note that with $\boldsymbol{v}_0 = 0$, density perturbation does not enter the expression of $\boldsymbol{j}_1$. It suffices to consider the momentum equation alone.

Let the perturbations be in the form of $\exp\left[i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)\right]$. The linearized momentum equation becomes

$$-i\omega \boldsymbol{v}_{1s} = \frac{q_s}{m_s} \left(\boldsymbol{E}_1 + \frac{\boldsymbol{v}_{1s}}{c} \times \boldsymbol{B}_0 \right) \;. \tag{20}$$

Unmagnetized modes

Without background magnetic fields, plasma response is simply given by

$$-i\omega \boldsymbol{v}_{1s} = \frac{q_s}{m_s} \boldsymbol{E}_1 \;, \tag{21}$$

The net plasma current is given by

$$\boldsymbol{j}_1 = \sum_s q_s n_{0s} \boldsymbol{v}_{1s} = \sum_s i \frac{n_s q_s^2}{\omega m_s} \boldsymbol{E}_1 \;. \tag{22}$$

Note that the phase of plasma current is offset from electric field. To find the dielectric tensor, we write

$$i \frac{4\pi}{\omega} \boldsymbol{j}_1 = -\sum_s \frac{4\pi}{\omega^2} \frac{n_s q_s^2}{m_s} \boldsymbol{E}_1 = -\sum_s \frac{\omega_{ps}^2}{\omega_p^2} \boldsymbol{E}_1 \;, \tag{23}$$

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[2]How cold is "cold"? Well, the answer can be problem dependent. The bottom line is, particle thermal velocity must be much smaller than the phase velocity of the wave of interest.

where ω_{ps} is the plasma frequency for species s. Because $\boldsymbol{j}_1 \parallel \boldsymbol{E}_1$, the dielectric tensor in this case is simply a scalar, given by

$$\epsilon = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2} ~, \tag{24}$$

where $\omega_p^2 \equiv \sum_s \omega_{ps}^2 \approx \omega_{pe}^2$.

Now we are ready to substitute ϵ back to (17) to find the dispersion relation. Without loss of generality, let us take $\boldsymbol{k}$ to be in the $\hat{z}$ direction, then (17) becomes

$$\begin{pmatrix} \epsilon - n^2 & 0 & 0 \\ 0 & \epsilon - n^2 & 0 \\ 0 & 0 & \epsilon \end{pmatrix} \begin{pmatrix} E_{1x} \\ E_{1y} \\ E_{1z} \end{pmatrix} = 0 ~, \tag{25}$$

where $\boldsymbol{n} \equiv \boldsymbol{k}c/\omega$ is called the *refractive index*.

Clearly, there are two types of modes. The dispersion relation for the first type is given by

$$\epsilon = 0 ~, \quad \text{or} \quad \omega^2 = \omega_p^2 ~. \tag{26}$$

This mode simply describes plasma oscillations, also known as *Langmuir waves*. Since no magnetic field is involved, it is an electrostatic mode. Also note that for this mode, only the $\hat{z}$-component of the electric field is non-zero as seen from (25), thus $\boldsymbol{k} \parallel \boldsymbol{E}_1$. This kind of wave is called a *longitudinal* wave. Langmuir waves have phase velocity $v_{\text{ph}} = \omega_p/k$, but the group velocity is zero: there is no energy flux associated with plasma oscillations.

For the second type of mode, the dispersion relation is given by

$$n^2 = \epsilon ~, \quad \text{or} \quad \omega^2 = \omega_p^2 + k^2 c^2 ~. \tag{27}$$

In this mode, we have $\boldsymbol{E}_1 \perp \boldsymbol{k}$. From (12), we also have $\boldsymbol{B}_1 = (\boldsymbol{k}c/\omega) \times \boldsymbol{E}_1$, and hence $\boldsymbol{B}_1 \perp \boldsymbol{k}$ as well. Therefore, this mode is an *electromagnetic* mode (which is a *transverse wave*), which is the counterpart of light waves in vacuum. Clearly, the dispersion relation demands that $\omega > \omega_p$: electromagnetic waves at frequency $\omega < \omega_p$ can not propagate through the plasma. We also note that for this mode, the phase velocity $v_{\text{ph}} > c$. However, it does not violate causality because information propagates at group speed, which is

$$v_g = \frac{\partial \omega}{\partial k} = \left(1 - \frac{\omega_p^2}{\omega^2} \right)^{1/2} c ~. \tag{28}$$

We also note that from (25), there are two electromagnetic modes, and they are degenerate. These properties are identical to light waves (e.g., linear/circular polarization). We finally comment that the degeneracy is broken when magnetic fields are included. In this case, eigen modes are circularly polarized modes, which propagate at different speeds, leading to a phenomenon known as *Faraday rotation*.

Low-frequency modes in a magnetized cold plasma

Including a background magnetic field makes the algebra much more complex. See textbook for details. Here, we focus on waves at frequencies much lower than the electron cyclotron frequency, and

the problem for evaluating plasma response is substantially simplified. Moreover, in this case, it suffices to ignore the displacement current (when doing so, we no longer use the dielectric tensor ϵ).

Recall that the electron equation of motion is

$$-i\omega m_e \boldsymbol{v}_e = -e\left(\boldsymbol{E}_1 + \frac{\boldsymbol{v}_e}{c} \times \boldsymbol{B}_0\right) . \tag{29}$$

In the low-frequency limit, we can ignore the electron inertia term on the left hand side. This means

$$\boldsymbol{E}_1 + \frac{\boldsymbol{v}_e}{c} \times \boldsymbol{B}_0 \approx 0 . \tag{30}$$

This is to be combined with the ion equation of motion

$$-i\omega m_i \boldsymbol{v}_{1i} = e\left(\boldsymbol{E}_1 + \frac{\boldsymbol{v}_{1i}}{c} \times \boldsymbol{B}_0\right) \approx e\left(\frac{\boldsymbol{v}_{1i} - \boldsymbol{v}_{1e}}{c}\right) \times \boldsymbol{B}_0 = \frac{\boldsymbol{j}_1 \times \boldsymbol{B}_0}{nc} . \tag{31}$$

At this point (after ignoring electron inertia and displacement current), we have essentially obtained the so-called Hall-MHD equations (in cold plasma limit): electrons are considered as a massless, neutralizing fluid, while ions carry all the inertia (so that fluid velocity is the ion velocity). The electron equation of motion expresses the generalized Ohm's law

$$\boldsymbol{E}_1 = -\frac{\boldsymbol{v}_{1e}}{c} \times \boldsymbol{B}_0 = -\frac{\boldsymbol{v}_{1i}}{c} \times \boldsymbol{B}_0 + \frac{\boldsymbol{j}_1 \times \boldsymbol{B}_0}{enc} , \tag{32}$$

where the $\boldsymbol{j}_1 \times \boldsymbol{B}$ term is the Hall term, characterizing the electron-ion drift.

Combine the the above two equations with the induction equation, and $\boldsymbol{j}_1 = (c/4\pi)i\boldsymbol{k} \times \boldsymbol{B}$, we obtain the linearized equations for cold plasma waves in the low-frequency limit

$$-\omega\rho\boldsymbol{v}_1 = \frac{1}{4\pi}(\boldsymbol{k} \times \boldsymbol{B}_1) \times \boldsymbol{B}_0 \tag{33}$$

where $\rho \approx nm_i$ is the fluid density.

$$-\frac{\omega}{c}\boldsymbol{B}_1 = \boldsymbol{k} \times \left[\frac{\boldsymbol{v}_1}{c} \times \boldsymbol{B}_0 - i\frac{(\boldsymbol{k} \times \boldsymbol{B}_1) \times \boldsymbol{B}_0}{4\pi en}\right] . \tag{34}$$

Now we consider waves propagating parallel to background magnetic field, with $\boldsymbol{k} = k\boldsymbol{e}_z$. In this case, we see that both $\boldsymbol{v}_1$ and $\boldsymbol{B}_1$ are perpendicular to $\boldsymbol{e}_z$: the resulting waves are transverse. The linearized equations reduce to

$$
\begin{aligned}
-\omega\rho\boldsymbol{v}_1 &= \frac{kB_0}{4\pi}\boldsymbol{B}_1 , \\
-\frac{\omega}{c}\boldsymbol{B}_1 &= \frac{kB_0}{c}\boldsymbol{v}_1 - i\frac{kB_0}{4\pi en}(\boldsymbol{k} \times \boldsymbol{B}_1) ,
\end{aligned}
\tag{35}
$$

and further

$$(\omega^2 - k^2 v_A^2)\boldsymbol{B}_1 = ik^2\omega\frac{B_0 c}{4\pi en}(\boldsymbol{e}_z \times \boldsymbol{B}_1) = ik^2 v_A^2 \frac{\omega}{\Omega_i}(\boldsymbol{e}_z \times \boldsymbol{B}_1) , \tag{36}$$

where $\Omega_i = eB_0/m_i c$ is the ion cyclotron frequency. Clearly, without the Hall term, we obtain the dispersion relation of the Alfvén waves. For parallel propagation, there are two degenerate Alfvén modes (shear and compressional). They can be decomposed into either two linear or two circular polarized

modes. One of them would become the slow/fast magnetosonic mode for oblique propagation (depending on plasma β). In the presence of the Hall term, however, the degeneracy is broken, and the eigen-vectors of the two modes correspond to circular polarized waves at different phase speeds. Written in matrix form, the above equation becomes

$$
\begin{pmatrix} \omega^2 - k^2 v_A^2 & ik^2 v_A^2 (\omega/\Omega_i) \\ -ik^2 v_A^2 (\omega/\Omega_i) & \omega^2 - k^2 v_A^2 \end{pmatrix} \begin{pmatrix} B_{1x} \\ B_{1y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} . \tag{37}
$$

The dispersion relation is obtained by requiring the determinant of the matrix on the left hand side to be zero. The result is

$$
\omega^2 - k^2 v_A^2 = \pm k^2 v_A^2 \frac{\omega}{\Omega_i} . \tag{38}
$$

When taking the + sign, the eigenvector has $B_x = -iB_y$. When combined with the factor $e^{i(kz-\omega t)}$, we can see that it corresponds to right polarization as viewed from the observer. This mode is called the electron whistler wave. When taking the $-$ sign, the eigenvector has $B_x = -iB_y$ corresponding to right polarization as viewed from the observer. This mode is called the ion cyclotron wave.

We can normalize the wave number by $x \equiv kv_A/\Omega_i = kd_i$ (note that we have the identity for the ion inertial length $d_i = c/\omega_{pi} = v_A/\Omega_i$). The dispersion relation then becomes

$$
\frac{\omega}{\Omega_i} = \frac{x}{2}(\sqrt{x^2 + 4} \pm x) . \tag{39}
$$

where again $\pm$ sign corresponds to right/left polarization.

For the ion cyclotron wave, we see that the dispersion relation is that $\omega \approx kv_A$ when $x \ll 1$, while $\omega \to \Omega_i$ as $x \to \infty$. In other words, the ion cyclotron wave can not propagate at frequencies higher than the ion cyclotron frequency. What happens at $\omega = \Omega_i$ is that the rotation of the electric vector matches the ion gyro-motion, leading to constant acceleration of the ions and rapid wave damping/absorption.

For the electron whistler wave, we see that at $x \gtrsim 1$, the dispersion relation is given by $\omega \propto k^2$. In other words, shorter-wavelength modes propagates faster. This mode is called the whistler wave because historically, it was first encountered by radio operators who heard, in their earphones, strange tones with rapidly changing pitch (descending). These turned out to be whistler modes excited by lightening in the southern hemisphere, that propagate along the Earths magnetic field to the northern hemisphere. The electron whistler wave, analogous to the ion cyclotron wave, can not propagate at frequencies higher than Ω_e, i.e., the electron cyclotron resonance. However, our approximation does not capture this resonance because we have ignored electron inertia and displacement current.

The general case

Well, there are too many different regimes. See the textbook for details. The main results are nicely summarized in the so-called Clemmow-Mullally-Allis (CMA) diagram, which shows the "wave-normal surfaces", which are polar plots of the wave phase velocity at fixed ω, as a function of obliquity θ (angle between $\boldsymbol{k}$ and $\boldsymbol{B}$). The attached CMA diagram is taken from Blandford & Thorne's online book, whose detailed figure caption is self-explanatory (in the plots, $\boldsymbol{B}$ is in the vertical direction). The two cases we discussed above correspond to the two regimes in the lower bottom and upper right corners.
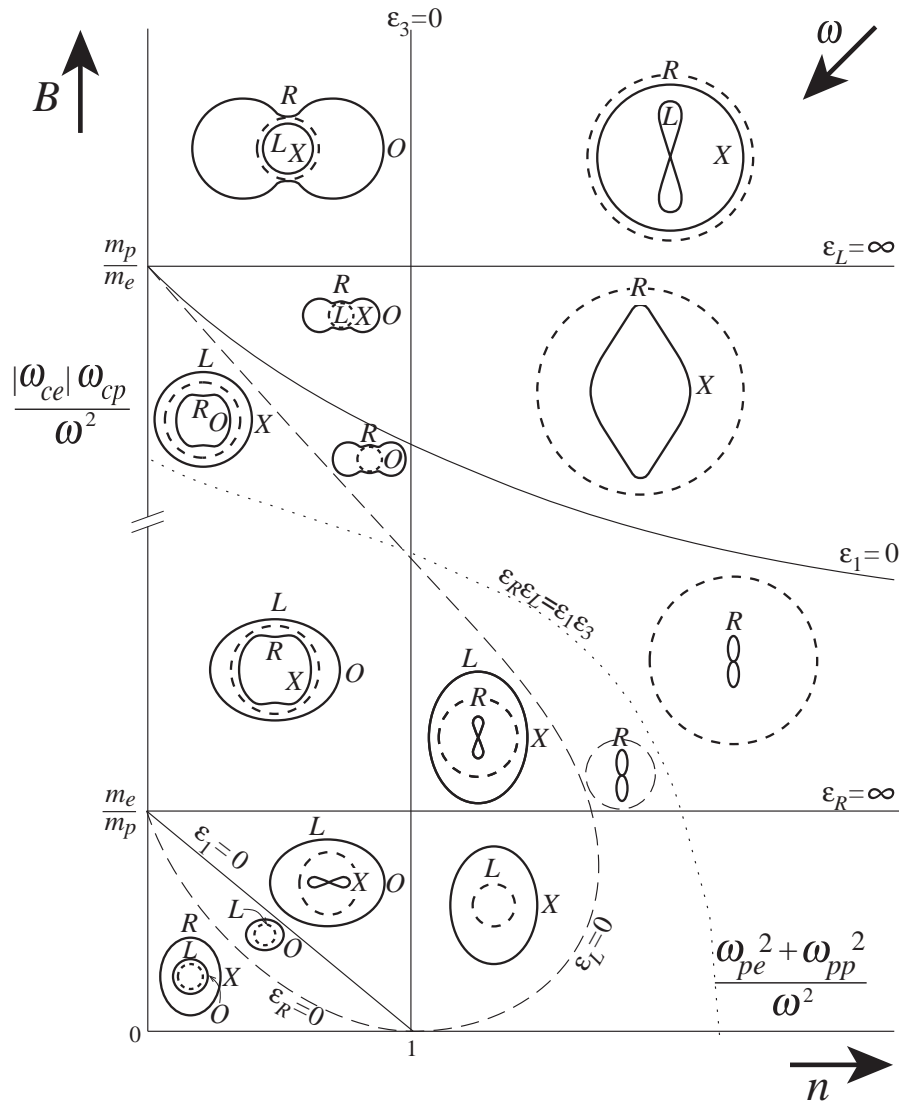
Fig. 21.7: Clemmow-Mullally-Allis (CMA) Diagram for wave modes with frequency ω propagating in a plasma with plasma frequencies ω_{pe}, ω_{pp} and gyro frequencies ω_{ce}, ω_{cp}. Plotted upward is the dimensionless quantity $|\omega_{ce}|\omega_{cp}/\omega^2$, which is proportional to B^2, so magnetic field strength also increases upward. Plotted rightward is the dimensionless quantity $(\omega_{pe}^2 + \omega_{pp}^2)/\omega^2$, which is proportional to n, so the plasma number density also increases rightward. Since both the ordinate and the abscissa scale as $1/\omega^2$, ω increases in the left-down direction. This plane is split into sixteen regions by a set of curves on which various dielectric components have special values. In each of the sixteen regions are shown two or one or no wave-normal surfaces (phase-velocity surfaces) at fixed ω; cf. Fig. 21.6a. These surfaces depict the types of wave modes that can propagate for that region's values of frequency ω, magnetic field strength B, and electron number density n. In each wave-normal diagram the dashed circle indicates the speed of light; a point outside that circle has phase velocity $V_{\rm ph}$ greater than c; inside the circle, $V_{\rm ph} < c$. The topologies of the wave normal surfaces and speeds relative to c are constant throughout each of the sixteen regions, and change as one moves between regions. [Adapted from Fig. 6.12 of Boyd and Sandersson (1973), which in turn is adapted from Allis, Buchsbaum and Bers (1963).]