Astronomy 253 (Spring 2016)
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Non-linear Effects

In the previous lecture, we discussed thermal effects and resonances on the linear properties of a plasma, and showed that resonant interaction leads to growth or damping of waves depending on the slope of the distribution function at resonances. In particular, we have already seen that there are many ways that a plasma can become unstable. These instabilities arise mainly because there are some forms of free energy available in the system, mainly residing in the distribution of particles. Growth of these instabilities are fed by such free energies, and of course should act to reduce them, which would eventually quench the instabilities. This means that particle distribution in phase space must be modified, but it is not captured in the linear analysis. To better understand the non-linear outcome of these instabilities, and assess to what extent are linear theories are valid, it is essential to move one step further to develop a theory applicable to the weakly non-linear regime.

Spectrum of waves

In all our previous study of waves, it suffices to decompose physical quantities into monochromatic modes, and then focus on a single mode with a fixed k. Non-linear theory inevitably involves coupling between different wave modes, so we must do the decomposition with more care. The foundation of this wave decomposition is a spatial Fourier transform inside a "box" of length L

$$E(k,t)=\int_{-L/2}^{L/2}E(x,t)e^{-\mathrm{i}kx}dx\ ,\quad E(x,t)=\frac{1}{L}\sum_k e^{\mathrm{i}kx}E(k,t)\ , \tag{1}$$

where $k=2\pi n/L$, $n=0,\pm1,\pm2,....$ Taking the limit $L\to\infty$ would convert the above pair of equations into the familiar Fourier integrals:

$$E(k,t)=\lim_{L\to\infty}\int_{-L/2}^{L/2}E(x,t)e^{-\mathrm{i}kx}dx\ ,\quad E(x,t)=\int_{-\infty}^{\infty}E(k,t)e^{\mathrm{i}kx}\frac{dk}{2\pi}\ . \tag{2}$$

Let us consider the "intensity" of the wave electric field by performing an average over length (or volume in 3D) of the system

$$\begin{aligned}\frac{1}{L}\int_{-L/2}^{L/2}dxE(x)^2&=\frac{1}{L^3}\int_{-L/2}^{L/2}dx\sum_{k,k'}e^{\mathrm{i}(k'-k)x}E^*(k)E(k')=\frac{1}{L^2}\sum_k E^*(k)E(k)\\&=\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{E^*(k)E(k)}{L}\equiv\int_{-\infty}^{\infty}I(k)dk\ .\end{aligned} \tag{3}$$

where we have essentially used Parseval's theorem. Here, $I(k)\equiv E^*(k)E(k)/2\pi L$ describes the intensity of the wave, and the wave spectral energy density is simply given by $I(k)/4\pi$ (split by half into field and particle energies). Note that the factor L is explicit in the definition to avoid divergence.

The key focus in weakly non-linear wave theory is to study the evolution of $I(k)$. This is described by the *wave-kinetic* equation, which is a very complex equation that essentially includes all the non-linear mode couplings and wave-particle interactions. Instead of presenting it, we here focus on the main ingredients of non-linear wave theory.

Wave-particle interaction: quasi-linear theory

Wave-particle interaction describes the long-term response of particle distribution function to waves. The standard theory for this is called quasi-linear theory (why we call it "quasi-linear" will become clear as we go through the derivations), which describes the diffusive evolution of the particle distribution function in velocity space.

Consider the simplest case of electrostatic waves in an unmagnetized plasma in 1D, and treat ions as neutralizing background. We separate the distribution function f, can be separated into a slowly-varying part, f_0, and a rapidly fluctuating part f_1 due to waves, where $f_1 \ll f_0$. For simplicity, we assume f_0 is spatially uniform.

$$f(x,v,t) = f_0(v,t) + f_1(x,v,t) \ . \tag{4}$$

The Vlasov equation reads

$$\frac{\partial f_0}{\partial t} + \frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial x} - \frac{e}{m}E\frac{\partial f_0}{\partial v} - \frac{e}{m}E\frac{\partial f_1}{\partial v} = 0 \ . \tag{5}$$

Averaging this equation over rapid fluctuations, we have

$$\frac{\partial f_0}{\partial t} = \frac{e}{m}\left\langle E\frac{\partial f_1}{\partial v}\right\rangle \ , \tag{6}$$

where $\langle\cdot\rangle$ denotes such time-averages. All other terms linear in f_1 or E average to zero. Note that the right hand side is a non-linear term and reflects the role of wave-particle interaction. Subtracting this part from the full Vlasov equation, we find the rapidly fluctuating part of the equation

$$\frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial x} - \frac{e}{m}E\frac{\partial f_0}{\partial v} = \frac{e}{m}\left(E\frac{\partial f_1}{\partial v} - \left\langle E\frac{\partial f_1}{\partial v}\right\rangle\right) \ . \tag{7}$$

As long as f_1 is much smaller compared with f_0, we can argue that the non-linear term on the right hand side is small compared with other linear terms and hence can be neglected. In this sense, the theory is quasi-linear: (6) is non-linear, while (7) is linear, which we replace by

$$\frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial x} - \frac{e}{m}E\frac{\partial f_0}{\partial v} = 0 \ . \tag{8}$$

This is very similar to the linearized Vlasov equations we used before but not quite the same, because f_0 is time-dependent. However, since the rate of change in f_0 is much slower compared with f_1, we may treat f_0 as constant in solving this equation.

Quasi-linear theory demands that first, higher-order processes should not be important, which is the case if the field amplitudes remain modest, and second, a sufficient number of modes are present so that phase-mixing is able to destroy, on the timescale of plasma-state evolution, any effects due to mode-mode coherence.

This equation for electrons may be coupled with the equation for other species, together with the Poisson equation, to yield a dispersion relation (following the procedures discussed in the previous lectures)

$$\omega = \omega(k) = \omega_R(k) + \mathrm{i}\gamma(k) \ . \tag{9}$$

Using this dispersion relation, we obtain

$$f_1(k,t) = \frac{\mathrm{i}e}{m}\frac{E(k,t)}{\omega(k)-kv}\cdot\frac{\partial f_0}{\partial v} \ , \tag{10}$$

which is to be substituted to (6) via a Fourier transformation.

To proceed, we invoke the so-called "random phase approximation". It states that for wave quantities in Fourier space $A(k,t)$ and $B(k,t)$, they satisfy $\langle A^*(k)B(k')\rangle = A^*(k)B(k')\delta(k,k')$. The quasi-linear term becomes

$$\begin{aligned}\frac{e}{m}\left\langle E\frac{\partial f_1}{\partial v}\right\rangle &= \frac{e}{m}\frac{1}{L^2}\sum_{k,k'} e^{\mathrm{i}(k-k')x}\left\langle E^*(k')\frac{\partial}{\partial v}f_1(k)\right\rangle \\ &= \frac{e}{m}\frac{1}{L^2}\sum_k E^*(k)\frac{\partial}{\partial v}f_1(k) = \frac{e}{m}\frac{1}{L}\int_{-\infty}^{\infty}\frac{dk}{2\pi}E^*(k)\frac{\partial}{\partial v}f_1(k) \ .\end{aligned} \tag{11}$$

Substituting (10) into the above, we find

$$\frac{e}{m}\left\langle E\frac{\partial f_1}{\partial v}\right\rangle = \mathrm{i}\frac{e^2}{m^2}\frac{1}{L}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{\partial}{\partial v}\left[\frac{E^*(k,t)E(k,t)}{\omega(k)-kv}\frac{\partial f_0}{\partial v}\right] = \frac{\partial}{\partial v}\left(D(v)\frac{\partial f_0}{\partial v}\right) \ , \tag{12}$$

where

$$D(v) \equiv \mathrm{i}\frac{e^2}{m^2}\frac{1}{L}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{E^*(k,t)E(k,t)}{\omega(k)-kv} = \mathrm{i}\frac{e^2}{m^2}\int_{-\infty}^{\infty}dk\frac{I(k)}{\omega(k)-kv} \tag{13}$$

is the diffusion coefficient in velocity space.

When $\gamma(k) \ll \omega_R(k)$, helpful insight can be gained by using the Plemelj relation[1]

$$\frac{1}{\omega - kv} = \mathcal{P}\left(\frac{1}{\omega_R - kv}\right) - \mathrm{i}\pi\delta(\omega_R - kv) \ . \tag{14}$$

Again, the principle value part describes contribution from non-resonant particles. Their oscillatory trajectories should not contribute to the changes in distribution function f_0 (but they retain memory of the waves, and contribute to energy and momentum conservation associated with the waves). The imaginary part describes contribution from resonant particles, which encapsulates the effect of particle redistribution following resonant interactions. Since the difference between ω and ω_R is small, we obtain

$$D(v) = \pi\frac{e^2}{m^2}\int_{-\infty}^{\infty}dkI(k)\delta(\omega(k)-kv) \ . \tag{15}$$

We see for particles at a given v, only waves that are resonant with these particles contribute to their velocity space diffusion.

[1]It again originates from causality considerations. A rule-of-thumb to obtain this relation is to replace ω by $\omega + \mathrm{i}\nu$ in the limit $\nu \to 0^+$.

Accompanying this equation of quasi-linear diffusion is the equation that determines the wave amplitude, which results from the solution to the linear dispersion relation. In terms of wave intensity (square of amplitude), we have

$$\frac{\partial}{\partial t} I(k) = 2\gamma(k) I(k) \ . \tag{16}$$

Note that in this equation, there can be additional terms due to other non-linear effects which are not included.

<u>Efficiency of quasi-linear diffusion</u>

How efficient is quasi-linear diffusion? We can proceed further by performing the integral over k by noting that

$$\delta(\omega(k) - kv) = \frac{\delta(k - k_c)}{|\partial\omega/\partial k - v_{\rm ph}|} = \frac{\delta(k - k_c)}{|v_g - v_{\rm ph}|} \ , \tag{17}$$

where k_c is the wavenumber such that $\omega(k) = kv$, and $v_{\rm ph}$ is the phase velocity at the resonance. Thus

$$D(v) = \pi \frac{e^2}{m^2} \frac{I(k)}{|v_g - v_{\rm ph}|}\bigg|_{\omega(k)=kv} \ . \tag{18}$$

For Langmuir waves, we have $v_{\rm ph} \approx \omega_p/k$, and $v_g \approx 3(k\lambda_D)v_e \ll v_{\rm ph}$ (for $k\lambda_D \ll 1$). If we consider a spectrum of waves around wavenumber k, the wave energy density is of the order $\mathcal{E} \approx kI(k)/4\pi$ (this is to assume that a wave packet has a size of $\sim 1/k$). If we assume this is a fraction δ of the thermal energy density nk_BT, we have

$$D(v) \approx \pi \frac{4\pi e^2}{m^2} \frac{\delta \cdot nk_BT}{\omega_{pe}} \approx \pi\delta v_e^2 \omega_{pe} \ . \tag{19}$$

We can see that diffusion timescale, the timescale over which particle velocity develops substantial spread (in this case $\Delta v \sim v_e$) is on the order of δ^{-1} times the plasma oscillation period. Thus, quasi-linear diffusion can be very efficient if wave amplitude is modestly large.

<u>Generalization to three dimensions</u>

In the case of electrostatic modes in unmagnetized plasmas, the dispersion relation is isotropic $\omega(k) = \omega(|\boldsymbol{k}|)$. Generalization of the discussions above is straightforward. Wave intensity is now defined as

$$\frac{1}{V}\int dV E(\boldsymbol{x})^2 = \int \frac{d^3k}{(2\pi)^3} \frac{E^*(\boldsymbol{k})E(\boldsymbol{k})}{V} \equiv \int I(\boldsymbol{k}) d^3k \ . \tag{20}$$

The diffusion coefficient becomes a tensor

$$\mathsf{D}(\boldsymbol{v}) = \pi \frac{e^2}{m^2} \int d^3k I(\boldsymbol{k}) \hat{k}\hat{k} \delta[\omega(\boldsymbol{k}) - \boldsymbol{k}\cdot\boldsymbol{v}] \ , \tag{21}$$

and the corresponding diffusion equation is

$$\frac{\partial f(\boldsymbol{v})}{\partial t} = \left(\frac{\partial}{\partial v_i} D_{ij} \frac{\partial}{\partial v_j}\right) f \ . \tag{22}$$

<u>Notes on random phase approximation</u>

It is well known that if the mean square of the Fourier transform is a smooth function of k, then the inverse transform is a series of wave packets. We expect these wave packets to fill the volume and each packet should have a random phase and position with respect to one another. Physically, we expect the wave packets to arise by exponential growth from a thermal distribution that satisfy these properties. This "random phase approximation" is a basic assumption in the quasi-linear theory of wave-particle interaction.

For simplicity, we treat the waves as one-dimensional. We consider N identical wave packets located in between $-L/2$ and $L/2$. Each packet is centered at $x = x_n$, and the electric field is given by

$$E^n = f(x - x_n, t) \ . \tag{23}$$

The total electric field is given by $E(x) = \sum_n E_n(x)$. Its Fourier transform is

$$E_n(k) = \int_{-L/2}^{L/2} f(x - x_n) e^{-\mathrm{i}kx} dx = e^{-\mathrm{i}kx_n} f(k) \ , \tag{24}$$

where $f(k)$ is the Fourier transform of $f(x)$. Because of the random phase, $\sum_n E_n(k)$ should average to zero. For wave intensity, we have

$$E^*(k')E(k) = \sum_{m,n} E_m^*(k')E_n(k) = \sum_{m,n} e^{\mathrm{i}(k'x_m - kx_n)} f^*(k')f(k) \ . \tag{25}$$

We then perform an ensemble average on the phases (x_n)

$$\begin{aligned} \langle E^*(k')E(k)\rangle =& \left\langle \sum_{m,n} e^{\mathrm{i}(k'x_m - kx_n)} \right\rangle f^*(k')f(k) \\ =& \frac{1}{L} \sum_n \int_{-L/2}^{L/2} dx_n e^{\mathrm{i}(k'-k)x_n} f^*(k')f(k) \\ =& \sum_n |f(k)|^2 \delta(k', k) = |E(k)|^2 \delta(k', k) = \frac{2\pi}{L} |E(k)|^2 \delta(k' - k) \ . \end{aligned} \tag{26}$$

Wave-wave interaction

Wave-wave interaction is essentially a fluid phenomenon and is best described in the fluid framework. To illustrate this, we may start from the continuity equation

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot [(\rho_0 + \rho_1)\boldsymbol{v}_1] = 0 \ , \tag{27}$$

where ρ_1 represents density perturbations. To first order, suppose we have two waves a and b given by

$$\rho_{1a} = \bar{\rho}_{1a} e^{\mathrm{i}(k_a x - \omega_a t)} \ , \quad \rho_{1b} = \bar{\rho}_{1b} e^{\mathrm{i}(k_b x - \omega_b t)} \ . \tag{28}$$

Now to the next order, we include the non-linear term $\nabla \cdot (\rho_1 \boldsymbol{v}_1)$. Note that for a single wave a or b, including the non-linear term leads to wave steepening and shocks that we discussed earlier in this course. The presence of two waves produces a cross term between them, which is what we are most interested. This cross term has the form

$$\nabla \cdot (\rho_1 \boldsymbol{v}_1) = \mathrm{i}(k_a + k_b)(\rho_{1b} v_{1a} + \rho_{1a} v_{1b}) \ , \tag{29}$$

which varies as $\exp\left[\mathrm{i}(k_a+k_b)x - \mathrm{i}(\omega_a+\omega_b)t\right]$. Clearly, if the wavenumber $k_{ab} \equiv k_a + k_b$, and the frequency $\omega_{ab} \equiv \omega_a + \omega_b$, satisfy a dispersion relation for a third wave, then this non-linear term provides a possibility for the three waves to couple with each other.

More generally, three waves a, b, and c can be non-linearly coupled if their wave numbers and frequencies satisfy

$$\boldsymbol{k}_c = \boldsymbol{k}_a + \boldsymbol{k}_b \ , \quad \omega_c = \omega_a + \omega_b \ . \tag{30}$$

This is known as a resonant triad. Of course, we can keep going to higher orders which would yield coupling among four, five, ..., waves, but as long as wave amplitudes are modest, these higher order couplings are much less significant than the resonant triad.

Wave-wave interactions come in two flavors. In the first case, two waves add up to produce the third wave. In the second case, one wave gives energy to two other waves through mutual interactions. The latter is also called mode decay.

Here we provide a simple example of wave-wave interaction that leads to mode decay. Consider a homogeneous medium with mean density ρ_0 and background field $\boldsymbol{B} = B_0\hat{z}$. For simplicity, treat the fluid as being isothermal with sound speed c_s. The Alfvén speed is thus $v_A = B_0/\sqrt{4\pi\rho_0}$. Let us assume the plasma is strongly magnetized with $c_s < v_A$.

Consider a modestly large amplitude Alfvén wave a propagating along the background field ($\hat{z}$), and is polarized in the $\hat{x}$ direction. Take its wave number k_a to be positive, and its frequency is $\omega_a = k_a v_A$. In addition, we assume there is a small amplitude Alfvén wave b, and a small amplitude sound speed c. Both propagate along $\hat{z}$, with wave numbers k_b, k_c (either positive or negative) and frequencies ω_b, ω_c (always positive). Assume the Alfvén wave b is also polarized in $\hat{x}$. It can be found that the selection rule can be satisfied in the case with $k_b < 0$ and $k_c > 0$:

$$k_a = -|k_b| + k_c \ , \quad k_a v_A = |k_b| v_A + k_c c_s \ , \tag{31}$$

which requires

$$\frac{k_a}{|k_b|} = \frac{v_A + c_s}{v_A - c_s} \ . \tag{32}$$

Note that we have assumed $v_A > c_s$ at the beginning. The outcome of this coupling is that a modestly strong Alfvén wave can decay by interacting with a counter-propagating Alfvén wave of longer wavelength and giving energy into this wave together with a forward-propagating sound wave with smaller wavelength. Details of the calculation can be found in Kulsrud's book. The decay rate scales as the square of the amplitude of the a wave times ω_a.

Non-linear Landau damping

There is another important non-linear process that involves coherent scattering of energy between two waves. In this process, two waves A and B interact with each other to produce a *beat wave*, and this beat wave resonantly interacts with particles traveling at the speed $(\omega_A - \omega_B)/(k_A - k_B)$. The outcome is that both the lower-frequency wave and the resonant particles gain energy at the expense of the higher-frequency wave. This process is called non-linear Landau damping.

As an example, consider two circularly polarized Alfvén waves propagating along the background magnetic field B_0 in the $\hat{z}$ direction, with wave numbers $k_{A,B}$ and frequencies $\omega_{A,B}$. The perturbed fields become

$$\delta B_x = b_A \cos\phi_A + b_B \cos\phi_B \ , \quad \delta B_y = b_A \sin\phi_A + b_B \sin\phi_B \ , \tag{33}$$

where, as before, $\phi_{A,B} = k_{A,B}x - \omega_{A,B}t$. Then the perpendicular field

$$(\delta b_\perp)^2 = b_A^2 + b_B^2 + 2b_A b_B \cos(\phi_A - \phi_B) \tag{34}$$

and the field strength to second order is

$$B = \sqrt{B_0^2 + \delta b_\perp^2} \approx B_0 + \frac{\delta b_\perp^2}{2B_0} = \text{const} + \frac{b_A b_B}{B_0} \cos(\phi_A - \phi_B) \ . \tag{35}$$

Recall that variation of field strength along the field leads to the magnetic mirror force acting on an ion particle is

$$\frac{d}{dt}(mv_\parallel) = -\mu \frac{\partial}{\partial z} B = \frac{mv_\perp^2}{2}(k_A - k_B)\frac{b_A b_B}{B_0^2} \sin(\phi_A - \phi_B) \equiv G \sin(\phi_A - \phi_B) \ . \tag{36}$$

This mirror force is exactly analogous to an oscillating electric field, with $G = eE_{\text{eff}}$. The ion particle will be in resonance with this beat wave when

$$v_z = \frac{\omega_A - \omega_B}{k_A - k_B} = v_A \ . \tag{37}$$

Here v_A is the Alfvén speed not to be confused with the A wave.

In our homework problem, you will be able to show that for particles traveling along an oscillatory electric field, the phase-averaged rate of energy change is given by

$$\begin{aligned} \dot{\mathcal{E}} = -\int \frac{\pi}{2} \frac{\omega e^2 E_{\text{eff}}^2}{mk^2} \frac{\partial f}{\partial v_z}\bigg|_{v_z = v_A} d^2 v_\perp &= -\int \frac{\pi}{2} \frac{v_A G^2}{m(k_A - k_B)} \frac{\partial f}{\partial v_z}\bigg|_{v_z = v_A} d^2 v_\perp \\ &= -\int d^2 v_\perp \frac{\pi}{8} m v_A v_\perp^4 \left(\frac{b_A b_B}{B_0^2}\right)^2 (k_A - k_B) \frac{\partial f}{\partial v_z}\bigg|_{v_z = v_A} \ . \end{aligned} \tag{38}$$

Clearly, ions gain energy if $\partial f/\partial v_z$ at $v_z = v_A$ is negative, which essentially always holds.

Where is the energy extracted from? Here we just state without proof that what happens is that, in quantum mechanics language, the higher frequency wave with frequency ω_A gives off an energy $\hbar\omega_A$, where some of it, $\hbar\omega_B$, ending up in wave B, and the difference going to the resonant particles. In other words,

$$-\frac{\dot{\mathcal{E}}_A}{\omega_A} = \frac{\dot{\mathcal{E}}_B}{\omega_B} \ . \tag{39}$$

where $\dot{\mathcal{E}}_{A,B}$ is the energy gain rate for the A, B waves. We thus have

$$-\dot{\mathcal{E}}_A = \dot{\mathcal{E}} + \dot{\mathcal{E}}_B \ , \tag{40}$$

or

$$\dot{\mathcal{E}} = -\dot{\mathcal{E}}_A - \dot{\mathcal{E}}_B = -\dot{\mathcal{E}}_A \left(1 - \frac{k_B}{k_A}\right) = -\dot{\mathcal{E}}_A \left(1 - \frac{k_B}{k_A}\right) \ . \tag{41}$$

Therefore, the damping rate for the A wave is

$$-\dot{\mathcal{E}}_A = \frac{k_A}{k_A - k_B}\dot{\mathcal{E}} = -\int d^2 v_\perp \frac{\pi}{8} m k_A v_A v_\perp^4 \left(\frac{b_A b_B}{B_0^2}\right)^2 \frac{\partial f}{\partial v_z}\bigg|_{v_z = v_A} \tag{42}$$

If we assume $v_A \ll c_s$, then we have

$$\frac{\partial f}{\partial v_z} \approx -\frac{v_A}{v_i^2} f(0, v_\perp) = -\frac{n_0}{(2\pi v_i^2)^{3/2}} \frac{v_A}{v_i^2} e^{-v_\perp^2/2v_i^2} \ , \tag{43}$$

where $v_i = \sqrt{kT/m_i}$ is the ion thermal speed. Which lead to

$$-\dot{\mathcal{E}}_A = -\sqrt{\frac{\pi}{2}}\omega_A \frac{v_A}{v_i} n_0 m v_i^2 \left(\frac{b_A b_B}{B_0^2}\right)^2 = -2\gamma_{NL}\mathcal{E}_A \ . \tag{44}$$

Noting that $\mathcal{E}_A = b_A^2/4\pi$ (equal amount of energy in the fields and in particle motion), we further obtain

$$\gamma_{NL} \approx \frac{1}{2}\sqrt{\frac{\pi}{2}}\frac{v_i}{v_A}\left(\frac{b_B}{B_0}\right)^2 \omega_A \ . \tag{45}$$

Non-linear Landau damping is an important saturation mechanism for unstable waves. Let γ be the linear growth rate for wave A. In general, when wave A is unstable, a family of waves with similar wave numbers are also unstable and share similar wave amplitudes as wave A. Therefore, for order of magnitude, we may simply replace b_B by b_A and write the damping rate as

$$\gamma_{NL} \approx \frac{1}{2}\sqrt{\frac{\pi}{2}}\frac{v_i}{v_A}\left(\frac{b_A}{B_0}\right)^2 \omega_A \equiv \alpha\mathcal{E}_A \ . \tag{46}$$

The saturation amplitude is thus given by

$$\mathcal{E}_A \sim \frac{2\gamma}{\alpha} \ . \tag{47}$$

We note that if the linear growth rate is very small, the wave can saturate at fairly low amplitude.

We finally comment that non-linear Landau damping operates only in collisionless plasmas, while damping (mode decay) by wave-wave interaction does not require this.

Particle trapping

Another non-linear mechanism for the saturation of a growing collisionless mode is particle trapping. We again take the electrostatic plasma oscillation as an example. In the wave frame, suppose the electrostatic potential varies as $\phi = \phi_0 e^{ikx}$, then in this frame, particles whose energy is less than $e\phi_0$ would get trapped and bounce back and forth in this electrostatic potential. This bounce motion is not captured in linear analysis, where particles are assumed to follow zeroth-order trajectories (straight lines). Because trapped particles do not exchange energy with the wave (note that in the non-linear regime, they are the resonant and are responsible for wave damping/growth), therefore, linear results of Landau damping/growth no longer hold. In the case of instabilities, particle trapping essentially leads to wave saturation.

Now let us discuss the condition for particle trapping. The equation of motion for trapped particles is

$$\frac{d^2 x}{dt^2} \approx -\frac{q}{m} k\phi_0 \sin kx \approx -\frac{q}{m} k E_0 x \ . \tag{48}$$

Thus, trapped particles bounce over a period of

$$\tau_b \sim 2\pi \sqrt{\frac{m}{qkE_0}} \ . \tag{49}$$

Linear analysis fails when this period is smaller than the damping/growth time scale.