Astronomy 253 (Spring 2016)

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Collisions and Transport

Coulomb Collision Rates

Consider a test particle – let it be an electron – interacting with background field particles – let them be protons for the moment. The test particle travels at speed v_e, and we treat protons as being at rest (because their velocities are typically much smaller). When the electron passes by a single proton, its trajectory is characterized by the *impact parameter* b, which is the distance of closest approach if it were not to be deflected. The electron will be scattered by the Coulomb field of the proton, a process sometimes called *Rutherford scattering*. If the deflection angle is small $\theta_D \ll 1$ (which is generally the case, as we shall see), then we can calculate its value using the *impulse approximation*: integrate the perpendicular impulse by the proton's Coulomb field exerted to the electron along its unperturbed trajectory. The deflection angle is determined by

$$m_e v_e \theta_D = m_e dv_y = \int_{-\infty}^{\infty} F_y(t) dt = \int_{-\infty}^{\infty} \frac{e^2}{b^2 + v_e^2 t^2} \frac{b}{\sqrt{b^2 + v_e^2 t^2}} dt = \frac{2e^2}{bv_e} \ , \tag{1}$$

thus

$$\theta_D = \frac{b_o}{b} \text{ for } b \gg b_o \ , \tag{2}$$

where

$$b_o \equiv \frac{2e^2}{mv_e^2} \ . \tag{3}$$

Note that this b_o is the same as r_c defined in the first handout for a thermal electron, and characterizes the scale below which Coulomb potential energy dominates particle kinetic energy. We can also see here that when $b \lesssim b_o$, the impulse approximation breaks down and the electron would be significantly deflected (by more than one radian).

Now we discuss the timescale for an individual electron to be deflectd by Coulomb scattering with protons. We consider the particle to have undergone a "collision" if its trajectory is deflected by of order one radian, at which point the particle's direction of motion is effectively randomized. This "collision" can be achieved either by a single scattering event, which requires the impact parameter $b \lesssim b_o$, or the cumulative effect of many small-angle scatterings. In the former case, the collision frequency ν_D (inverse of collision time t_D) is

$$\nu_D \equiv \frac{1}{t_D} = n\sigma v_e = n\pi b_o^2 v_e \ , \tag{4}$$

where n is the proton number density.

In fact, the effect of cumulative small-angle scattering in a plasma is much more significant. Because the directions of individual scatterings are random, the deflection angle undergoes a random-walk. Hence,

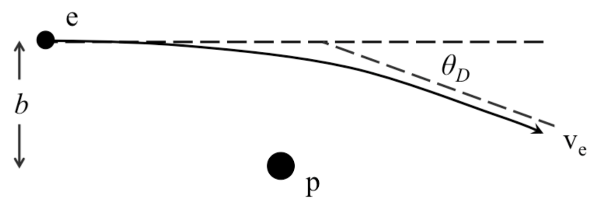

Figure 1: The geometry of a Coulomb interaction.

the mean deflection angle vanishes but its mean square, $\langle\Theta^2\rangle \equiv \sum_{\text{all encounters}} \theta_D^2$, does not. We can evaluate this by

$$\langle\Theta^2\rangle = \int_{b_{\min}}^{b_{\max}} \left(\frac{b_o}{b}\right)^2 n v_e t \cdot 2\pi b db = 2\pi n b_o^2 v_e t \ln\left(\frac{b_{\max}}{b_{\min}}\right) . \tag{5}$$

Here, the factor $(b_o/b)^2$ is the squared deflection angle, and the remaining factor $nv_e t 2\pi b db$ is the number of encounters that occur with impact parameters between b and $b + db$ during time t.

The integral diverges logarithmically at both lower limit $b_{\min}$ and upper limit $b_{\max}$. We will discuss the origins of and values of the cutoffs $b_{\min}$ and $b_{\max}$ below. The value of t for $\langle\Theta^2\rangle$ to reach unity defines the deflection time t_D

$$\nu_D = \frac{1}{t_D} = 2\pi n b_o^2 v_e \ln\Lambda = \frac{8\pi n e^4}{m^2 v_e^3} \ln\Lambda \ , \tag{6}$$

where $\Lambda \equiv b_{\max}/b_{\min}$. The factor $\ln\Lambda$ is called Coulomb logarithm. We see that this deflection frequency is larger, by a factor $2\ln\Lambda$ than the frequency for a single large-angle scattering.

The lower limit $b_{\min}$ should be taken as b_o below which the impact approximation breaks down [and we already take care of this regime in (4)][1]. The upper limit $b_{\max}$ should be taken as the Debye length λ_D, because the Coulomb force is largely shielded beyond λ_D. In the first lecture, we introduced the plasma parameter Λ. From (12) in the first handout, we see that the plasma parameter is essentially the same as Λ in the definition of Coulomb logarithm (hence the same symbol is used), if the electron velocity v_e we just considered is taken to be thermal. As we already learned that Λ is huge for typical astrophysical plasmas ($\gtrsim 10^5$), and the logarithmic form of this factor means that it does not depend on Λ very strongly. For Λ in the range of 10^{5-13}, $\ln\Lambda$ varies between about $10 - 30$. For general purposes, it suffices to take $\ln\Lambda \sim 20$. Clearly, interaction with many distant particles within the Debye sphere largely dominates close encounters.

The discussion above applies to electrons being scattered by ions. We can also consider the deflection of electrons by other electrons. Although we can no longer assume other electrons being at rest, the two-body interaction is equivalently described by a charged particle with reduced mass $\mu = m_e/2$ interacting with another charged particle at rest (plus other corrections such as relative velocity, etc.), and overall the corrections are of order unity. Therefore, the deflection frequency for electrons are simply given by

$$\nu_D^{ee} \sim \nu_D^{ep} \sim \frac{8\pi n e^4}{m_e^2 v_e^3} \ln\Lambda \ . \tag{7}$$

[1]Note that when b_o is smaller than the de Broglie wavelength of the electron, we need to replace b_o by $\hbar/m_e v_e$. This occurs when electron velocity $v_e \gtrsim 4.4 \times 10^8$ cm s^{-1}. The same applies to protons.

In the same spirit, we can compute the collision frequency/time for protons. Because protons are much more massive than the electrons, deflection is dominated by proton-proton collisions. The deflection frequency is given by

$$\nu_D^{pp} \sim \frac{8\pi n e^4}{m_p^2 v_p^3} \ln\Lambda \ . \tag{8}$$

Clearly, if electrons and protons share the same temperature T, then proton-proton collision rate is a factor $\sqrt{m_e/m_p}$ slower.

Thermal Equilibration Rates

Collisions of electrons with one another randomize their velocities at the rate of ν_D^{ee}. This can also be considered as the rate to establish a Maxwellian distribution. Similarly, collisions of protons with one another drives them to achieve their own Maxwellian distribution at the rate of ν_D^{pp}, which is a factor of $\sqrt{m_e/m_p}$ slower. If electrons and protons have different temperatures, interactions between electrons and ions will lead to exchange energy between the two species, and drive them towards a common temperature.

We again start from binary interactions. To analyze the energy exchange ΔE, it is most convenient to work in the proton frame. By momentum conservation, we have

$$\Delta E = -\frac{\delta p^2}{2m_p} = -\frac{m_e}{m_p}\left(\frac{b_o}{b}\right)^2 E \ \text{for}\ b \gg b_o \ . \tag{9}$$

Here $E = m_e v_e^2/2$ is the electron's initial energy. Compare this with the deflection rate (in θ_D^2) studied before, we see that $\Delta E/E$ is a factor m_e/m_p smaller. Correspondingly, the timescale for electron energy loss is

$$\nu_{\rm loss} \sim \frac{m_e}{m_p}\nu_D^{ep} \ . \tag{10}$$

Clearly, this is another factor of $\sqrt{m_e/m_p}$ slower than ν^{ep}, and is the slowest process among all. We also note that in arriving at (9), we assume that ions stand as static background, and hence electrons always lose energy to protons. When ions have finite temperature, electrons can either gain or lose energy from the ions during a Coulomb interaction, and (10) represents an order-of-magnitude estimate for the timescale for electrons and ions to equilibrate.

Collision Times from the Fokker-Planck Equation

The collision rates above are obtained in a qualitative way. More accurate result can be derived from the Fokker-Planck equation that will be discussed in more detail. For Maxwellian distributions at temperature T, we list τ_{AB} $(= 1/\nu_D^{AB})$, the effective collision time of particle species A by interaction with particle species B, where A, B can be e or p:

$$\tau_{ep} = \frac{3}{4\sqrt{2\pi}\ln\Lambda}\frac{(kT)^{3/2}m_e^{1/2}}{e^4 n} \approx 2.8\times10^4\left(\frac{10}{\ln\Lambda}\right)\left(\frac{T}{10^4\ \rm K}\right)^{3/2}\left(\frac{n}{\rm cm^{-3}}\right)^{-1} \rm s \ , \tag{11}$$

$$\tau_{ee} = \frac{3}{4\sqrt{\pi}\ln\Lambda}\frac{(kT)^{3/2}m_e^{1/2}}{e^4 n} \approx 3.9\times10^4\left(\frac{10}{\ln\Lambda}\right)\left(\frac{T}{10^4\ \rm K}\right)^{3/2}\left(\frac{n}{\rm cm^{-3}}\right)^{-1} \rm s \ . \tag{12}$$

$$\tau_{pp} = \frac{3}{4\sqrt{\pi}\ln\Lambda}\frac{(kT)^{3/2}m_p^{1/2}}{e^4 n} \approx 1.7\times10^6\left(\frac{10}{\ln\Lambda}\right)\left(\frac{T}{10^4\ \rm K}\right)^{3/2}\left(\frac{n}{\rm cm^{-3}}\right)^{-1} \rm s \ . \tag{13}$$

$$\tau_{pe} = \frac{3}{4\sqrt{2\pi}\ln\Lambda}\frac{(kT)^{3/2}m_p}{e^4 n m_e^{1/2}} \approx 5.0\times 10^7\left(\frac{10}{\ln\Lambda}\right)\left(\frac{T}{10^4\ \mathrm{K}}\right)^{3/2}\left(\frac{n}{\mathrm{cm}^{-3}}\right)^{-1}\ \mathrm{s}\ . \tag{14}$$

They can be considered as the timescale for the particles to establish a Maxwellian distribution. As already discussed, the relative ordering is given by

$$\tau_{ee} \sim \tau_{ep} \sim (m_e/m_p)^{1/2}\tau_{pp} \sim (m_e/m_p)\tau_{pe}\ . \tag{15}$$

Electrons are the most easily scattered, ion scattering rate is a factor $(m_e/m_p)^{1/2}$ smaller, while scattering of the ions by the electrons is the most inefficient.

Recall that the Coulomb logarithm is essentially the same as the plasma parameter

$$\Lambda \approx \frac{(kT)^{3/2}}{(4\pi n)^{1/2}e^3}\ . \tag{16}$$

It is intuitive to find that

$$\nu_{ee} \equiv \frac{1}{\tau_{ee}} \sim \frac{\ln\Lambda}{\Lambda}\omega_{pe}\ , \quad \nu_{pp} \equiv \frac{1}{\tau_{pp}} \sim \frac{\ln\Lambda}{\Lambda}\omega_{pi}\ . \tag{17}$$

Knowing the collision frequency, it is straightforward to compute the particle mean free path $\lambda_{\rm mfp} \sim v_T\tau$, where $v_T^2 = 3kT/m$ is the particle thermal velocity. The result is

$$\lambda_e \sim v_e\tau_{ee} \sim \frac{3\sqrt{3}}{4\sqrt{\pi}\ln\Lambda}\frac{(kT)^2}{e^4 n} \sim \lambda_p \approx 2.6\times 10^{12}\left(\frac{10}{\ln\Lambda}\right)\left(\frac{n}{\mathrm{cm}^{-3}}\right)^{-1}\left(\frac{T}{10^4\mathrm{K}}\right)^2\ \mathrm{cm}\ . \tag{18}$$

Note that the ion and electron mean free paths are about the same if their temperatures are about equal.

Fokker-Planck equation

Following the overview of binary collision processes, we now return to the Boltzmann equation, with the addition of collisions

$$\frac{\partial f_s}{\partial t} + \boldsymbol{v}\cdot\nabla f_s + \left[\frac{q_s}{m_s}\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c}\times\boldsymbol{B}\right)\right]\cdot\frac{\partial f_s}{\partial \boldsymbol{v}} = \left(\frac{\partial f_s}{\partial t}\right)_c\ , \tag{19}$$

where the last term denotes "collisions" due to interparticle interactions, and the subscript denotes individual species s $(= i, e)$.

As we have seen, collisions in plasmas are largely due to the cumulative effect of many relatively weak interactions rather than few close encounters. In other words, these interactions only slightly alter particle trajectories in phase space, rather than giving discontinuous kicks. The Fokker-Planck equation describes the evolution of the distribution function under the influence of such weak and random interactions.

We define a function $\psi(\boldsymbol{v}, \Delta\boldsymbol{v}, \Delta t)$, so that

$$\psi(\boldsymbol{v}, \Delta\boldsymbol{v}, \Delta t)d^3\boldsymbol{v} \tag{20}$$

is the probability that a particle with velocity $\boldsymbol{v}$ be scattered to velocity $\boldsymbol{v} + \Delta\boldsymbol{v}$ within $d^3\boldsymbol{v}$ after a time interval Δt. Clearly, it must satisfy

$$\int \psi(\boldsymbol{v}, \Delta\boldsymbol{v}, \Delta t)d^3\Delta\boldsymbol{v} = 1\ . \tag{21}$$

With this definition, the evolution of the distribution function due to collisions after time Δt is given by

$$f(\boldsymbol{v}, t+\Delta t) = \int f(\boldsymbol{v}-\Delta\boldsymbol{v}, t)\psi(\boldsymbol{v}-\Delta\boldsymbol{v}, \Delta\boldsymbol{v}, \Delta t) d^3\Delta\boldsymbol{v} \ . \tag{22}$$

Because individual collisions are weak, most contribution in the integral above must come from small $\Delta\boldsymbol{v}$, allowing us to expand the equation around $\boldsymbol{v}$. Up to the second order, we have

$$f(\boldsymbol{v}, t+\Delta t) = f(\boldsymbol{v}, t) - \int \Delta\boldsymbol{v}\cdot\frac{\partial}{\partial\boldsymbol{v}}(f\psi)d^3\Delta\boldsymbol{v} + \frac{1}{2}\int \Delta\boldsymbol{v}\cdot\left[\Delta\boldsymbol{v}\cdot\frac{\partial}{\partial\boldsymbol{v}}\frac{\partial}{\partial\boldsymbol{v}}(f\psi)\right]d^3\Delta\boldsymbol{v} \ . \tag{23}$$

We can move the partial derivative on $\boldsymbol{v}$ out and perform the integral on $\Delta\boldsymbol{v}$, defining the following

$$\langle\dot{\overline{\Delta\boldsymbol{v}}}\rangle \equiv \frac{1}{\Delta t}\int \Delta\boldsymbol{v}\psi(\boldsymbol{v}, \Delta\boldsymbol{v}, \Delta t)d^3\Delta\boldsymbol{v} \ , \tag{24}$$

$$\langle\dot{\overline{\Delta\boldsymbol{v}\Delta\boldsymbol{v}}}\rangle \equiv \frac{1}{\Delta t}\int \Delta\boldsymbol{v}\Delta\boldsymbol{v}\psi(\boldsymbol{v}, \Delta\boldsymbol{v}, \Delta t)d^3\Delta\boldsymbol{v} \ . \tag{25}$$

Note that both $\langle\dot{\overline{\Delta\boldsymbol{v}}}\rangle$ and $\langle\dot{\overline{\Delta\boldsymbol{v}\Delta\boldsymbol{v}}}\rangle$ are functions of $\boldsymbol{v}$, and describes the rate of change in $\Delta\boldsymbol{v}$ and $\Delta\boldsymbol{v}\Delta\boldsymbol{v}$ due to collisions. Then, we arrive at the Fokker-Planck equation

$$\left(\frac{\partial f}{\partial t}\right)_c = -\frac{\partial}{\partial\boldsymbol{v}}\cdot(f\langle\dot{\overline{\Delta\boldsymbol{v}}}\rangle) + \frac{1}{2}\frac{\partial}{\partial\boldsymbol{v}}\cdot\left[\frac{\partial}{\partial\boldsymbol{v}}\cdot f\langle\dot{\overline{\Delta\boldsymbol{v}\Delta\boldsymbol{v}}}\rangle\right] \ . \tag{26}$$

Physical interpretation of the two terms on the right hand side is straightforward. The first term is the average rate of change of the particles' mean velocity due to collisions, and it is called *dynamical friction* and is analogous to the same phenomenon in the context of collisionless N-body dynamics. The second term is called velocity diffusion coefficient, which describes the spreading of particle velocity distribution.

The Landau Collision Integral

For plasmas with multiple species, the collision term for species A can be expressed as a summation of collision terms of A with all species (including itself), represented by Bs. For notational convenience, the collision term is often written as

$$\left(\frac{\partial f_A}{\partial t}\right)_c = \sum_B C_{AB}(f_A, f_B) \ , \tag{27}$$

where C_{AB} is the collision operator describing the rate of change in f_A due to interactions with particles of species B with distribution function f_B.

With a more careful derivation of Coulomb collisions and some algebra, the Fokker-Planck equation can be cast into a very physically intuitive form, given by

$$\left(\frac{\partial f_A}{\partial t}\right)_c = \frac{1}{m_A}\frac{\partial}{\partial\boldsymbol{v}_A}\cdot\left[\sum_B 2\pi q_A^2 q_B^2 \ln\Lambda \int f_A(\boldsymbol{v}_A)f_B(\boldsymbol{v}_B)\frac{\mathsf{I}-\hat{\boldsymbol{u}}\hat{\boldsymbol{u}}}{u}\cdot\boldsymbol{\chi}_{AB}d^3\boldsymbol{v}_B\right] , \tag{28}$$

where $\boldsymbol{u} \equiv \boldsymbol{v}_A - \boldsymbol{v}_B$ with $\hat{\boldsymbol{u}}$ being the unit vector along $\boldsymbol{u}$, and

$$\boldsymbol{\chi}_{AB} \equiv \frac{1}{m_A}\frac{\partial \ln f_A(\boldsymbol{v}_A)}{\partial\boldsymbol{v}_A} - \frac{1}{m_B}\frac{\partial \ln f_B(\boldsymbol{v}_B)}{\partial\boldsymbol{v}_B} \ . \tag{29}$$

This is known as the *Landau collision integral*, and the corresponding C_{AB} is called the *Landau collision operator*.

As long as collisions are elastic, the collision operator must satisfy conservation of particle number, momentum and energy. These can be easily demonstrated using the Fokker-Planck equation in Landau form, given its highly symmetric nature. In addition, it can be shown that the collision integral satisfies the Boltzmann *H-theorem*

$$\frac{dH}{dt} \leq 0 \ , \quad \text{where} \quad H \equiv \sum_A \int f_A \ln f_A d^3\boldsymbol{v}_A \ . \tag{30}$$

The quantity H is closely related to the entropy S of the system, with $S = -H + C$ where C is a constant, and the *Boltzmann H-theorem* reinforces the second law of thermodynamics.

It can further be shown that the condition for $dH/dt = 0$ is that all particles species follow a Maxwell distribution with a common temperature (e.g., Hinton, 1983). Therefore, Coulomb collisions drive the distribution functions of all particle species towards Maxwellian distribution.

Because of this property, it is often convenient to consider the simplest *Bhatnagar-Gross-Krook (BGK)* collision operator. For species s, it is given by

$$\left(\frac{\partial f_s}{\partial t}\right)_c = -\nu_s(f_s - F_{M,s}) \ , \tag{31}$$

where $F_{M,s}$ is the distribution function at equilibrium (Maxwellian), and ν_s is the collision frequency. Although it misses certain aspects of physics, its simplicity greatly facilitates analysis in many physical situations.

Transport Coefficients

Collisions are fundamental in understanding the plasma transport processes, including thermal conduction, viscosity, and resistivity. The transport coefficients can be derived rigorously from the Fokker-Planck equation, but they can be obtained to within order of magnitude from simple analysis, as we proceed below.

Transport coefficients in the absence of magnetic fields

For any species s (which we drop the subscript label) with number density n at temperature T (with thermal velocity $v_T \sim \sqrt{kT/m}$) , let ν be the collision time and collision frequency as derived above, and $\tau \equiv 1/\nu$ be the collision time. The general scaling is that $\tau \propto m^{1/2}T^{3/2}/n$.

Resistivity results from the collisional drag between the electrons and ions. Its inverse, electric conductivity σ, is defined by the Ohm's law

$$\boldsymbol{J} = \sigma \boldsymbol{E} \ , \tag{32}$$

where for the moment, we only consider the component parallel to the magnetic field. Imposing an external electric field E, a current develops as electrons and ions are accelerates to opposite directions, leading to a mean drift velocity v_d between the two species, with $v_d \sim J/en$. For sufficiently small drift velocity $v_d \ll v_T$, the collisional drag force F_d is proportional to v_d, with stopping time $\sim$ collision time τ:

$$F_d \sim \frac{mv_d}{\tau} = \frac{mJ}{en\tau} \ . \tag{33}$$

Balancing the parallel acceleration by the electric field and collisional drag force, electric conductivity is given by

$$\sigma \sim \frac{ne^2\tau}{m} \propto \frac{T^{3/2}}{m^{1/2}} \ . \tag{34}$$

The Fokker-Planck result is a factor of ~ 2 larger. We see that parallel electric conductivity becomes progressively larger towards higher temperature, and again, contribution from the most mobile electrons dominates that from the ions by a factor of $\sim \sqrt{m_i/m_e}$.

A heat flux $\boldsymbol{q}$ is generated in presence of a temperature gradient, which defines the thermal conductivity κ

$$\boldsymbol{q} = -\kappa \nabla T \ . \tag{35}$$

Microscopically, heat transport takes place over the distance of particle mean free path $\lambda_{\rm mfp} \sim v_T\tau$, where v_T is the thermal velocity, τ is the collision time. Particles can exchange energy with their neighbors by collisions at a distance of $\sim \lambda_{\rm mfp}$. The amount of energy exchange over one mutual collision, averaged over all particles, is $\Delta E \sim k\nabla T\lambda_{\rm mfp}$. The heat flux is thus given by $q \sim n\Delta E v_T \sim nk\nabla T\lambda_{\rm mfp}v_T$. The thermal conductivity is thus given by

$$\kappa \sim (nk)\lambda_{\rm mfp}v_T \sim \frac{n\tau kT}{m} \propto \frac{T^{5/2}}{m^{1/2}} \ . \tag{36}$$

The Fokker-Planck result is a factor of ~ 3 larger. We see that thermal conduction becomes progressively more important towards more weakly collisional systems and sensitively depends on temperature. It is also clear that electrons dominate heat conduction over the ions by a factor of $\sim \sqrt{m_i/m_e}$.

Viscosity can be considered as a similar phenomenon as thermal conduction: particles exchange of momentum (instead of energy) over the distance of the mean free path. The momentum flux, now is a tensor (called viscous stress tensor), is proportional to the velocity gradient. The formal expression of the viscous stress tensor is given by

$$\pi_{ij} = \eta\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\nabla\cdot\boldsymbol{V}\right) \ . \tag{37}$$

The dynamical viscosity η can be estimated in the same spirit as for thermal conductivity. The amount of momentum exchange over one collision is about $\Delta p \sim m\lambda_{\rm mfp}(dV/dl)$, leading to a momentum flux of $\pi \sim n\Delta p v_T \sim nm\lambda_{\rm mfp}(dV/dl)v_T$.

$$\eta \sim (nm)\lambda_{\rm mfp}v_T \sim n\tau kT \sim m^{1/2}T^{5/2} \ . \tag{38}$$

Because the momentum of the plasma is mostly carried by the ions, viscous stress is dominated by the ions by a factor of $\sqrt{m_i/m_e}$ over the electrons.

Transport coefficients in the presence of magnetic fields

The relative importance of magnetic fields is measured by the ratio of Larmor radius r_L and particle mean free path $\lambda_{\rm mfp}$. When $r_L \lesssim \lambda_{\rm mfp}$, particle motion perpendicular to the magnetic field is restricted. The effective mean free path perpendicular to the magnetic field essentially becomes $\lambda_{{\rm mfp},\perp} \sim r_L$, and the particle flux in the perpendicular direction is r_L/τ (instead of v_T).

We can then follow the same analysis above to obtain for the perpendicular thermal conductivity

$$\kappa_\perp \sim (nk) r_L \left(\frac{r_L}{\tau}\right) \sim \left(\frac{r_L}{\lambda_{\rm mfp}}\right)^2 \kappa_\parallel = \frac{\kappa_\parallel}{(\Omega_L \tau)^2} \ , \tag{39}$$

where Ω_L is the Larmor frequency. When $\Omega_L \gg \nu$, we see that perpendicular heat flux is strongly suppressed by a factor of $(\nu/\Omega_L)^2$, and hence thermal conduction proceeds primarily along the magnetic field.

Similarly, viscosity can be divided into parallel and perpendicular viscosity, though the corresponding stress tensor is much more complex, which we postpone to the next lecture. The same scaling holds between parallel and perpendicular viscosity, with $\eta_\perp \sim \eta_\parallel/(\Omega_L\tau)^2$.

In addition to the diffusive particle flux that leads to the diffusive transport of heat and momentum, there is also a transport process associated with the *gyro-magnetic particle flux*. The origin of gyro-magnetic flux is analogous to the *magnetization current* discussed in particle orbit theory. This effect gives a heat flux

$$\boldsymbol{q}_\times \sim n v_T (r_L \boldsymbol{b} \times \nabla kT) \ , \tag{40}$$

which is perpendicular to both the direction of the magnetic field and temperature gradient. The corresponding coefficient, the *cross thermal conductivity* $\kappa_\times$, is

$$\kappa_\times \sim n v_T k r_L \sim \frac{nkT}{m\Omega_L} \ . \tag{41}$$

The more accurate Fokker-Planck result is a factor 5/2 larger. Note that this this conductivity is independent of collision frequency, and its value is intermediate between $\kappa_\parallel$ and $\kappa_\perp$ ($\kappa_\times \sim \sqrt{\kappa_\parallel \kappa_\perp}$). In addition, since $\boldsymbol{q}_\times$ is perpendicular to ∇T, this heat flux is along isotherms and does not change the temperature of the system. There is also an analogous stress term for viscosity, which is called *gyro-viscosity*, though it is not a real viscosity because the gyro-viscous stress does not lead to any dissipation (viscous heating).

Perpendicular electric conductivity is obtained somewhat differently. Recall that electric conductivity is simply associated directly with the collisional drag force F_d. In the parallel case, we can rewrite the definition of conductivity as

$$\frac{J_\parallel}{\sigma_\parallel} = \frac{F_{d,\parallel}}{e} \ , \tag{42}$$

which can be directly generalized to the perpendicular direction. Since the drag force has the same form in the parallel and perpendicular cases

$$F_\parallel \sim \frac{m v_{d,\parallel}}{\tau} = \frac{m J_\parallel}{en\tau} \ , \quad F_\perp \sim \frac{m v_{d,\perp}}{\tau} = \frac{m J_\perp}{en\tau} \ , \tag{43}$$

we then find

$$\sigma_\perp \sim \sigma_\parallel \sim \frac{ne^2\tau}{m} \ . \tag{44}$$

More careful analysis using the Fokker-Planck equation gives $\sigma_\perp \approx \sigma_\parallel/2$. Electric conductivity feeds to the generalized Ohm's law discussed earlier in the course, which can be written as

$$\boldsymbol{E} + \frac{\boldsymbol{V}_e \times \boldsymbol{B}}{c} = ne\left(\frac{\boldsymbol{J}_\parallel}{\sigma_\parallel} + \frac{\boldsymbol{J}_\perp}{\sigma_\perp}\right) - \frac{\nabla \cdot \mathbf{P}_e}{en} \ , \tag{45}$$

where $\boldsymbol{V}_e$ and $\mathbf{P}_e$ are the electron fluid velocity and the electron pressure tensor.