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Fluid Closure and Kinetic MHD

The collisional Boltzmann equation for species s $(= i, e)$ reads

$$\frac{\partial f_s}{\partial t} + \boldsymbol{v} \cdot \nabla f_s + \left[\frac{q_s}{m_s}\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}\right)\right] \cdot \frac{\partial f_s}{\partial \boldsymbol{v}} = \sum_{s'} C_{ss'}(f_s, f_{s'}) \equiv \mathcal{C}[f_s] \ , \tag{1}$$

where $\mathcal{C}[f_s]$ is the collision operator, and incorporate contributions from collisions with all other species s'. Fluid equations can be systematically obtained by taking moments from this equation.

Moments of the Distribution Function

The kth moment of the distribution function f_s has the form of

$$\mathbf{M}_k(\boldsymbol{r}, t) \equiv \int \underbrace{\boldsymbol{v}\boldsymbol{v}...\boldsymbol{v}}_{k} f_s(\boldsymbol{r}, \boldsymbol{v}, t) d^3\boldsymbol{v} \ , \tag{2}$$

which is a tensor of rank k. The set of $\{\mathbf{M}_k, k = 0, 1, 2, ...\}$ can be viewed as an alternative representation of the distribution function, which uniquely specifies f_s if it is sufficiently smooth.

The zeroth moment of f_s simply gives particle number density

$$n_s(\boldsymbol{r}, t) = \int f_s(\boldsymbol{r}, \boldsymbol{v}, t) d^3\boldsymbol{v} \ . \tag{3}$$

The first moment gives particle flux density, from which we can define *flow velocity*

$$\boldsymbol{V}_s(\boldsymbol{r}, t) = \frac{1}{n_s} \int \boldsymbol{v} f_s(\boldsymbol{r}, \boldsymbol{v}, t) d^3\boldsymbol{v} \ . \tag{4}$$

The second moment describes the flow of momentum, and gives the *stress tensor*

$$\mathbf{T}_s(\boldsymbol{r}, t) = \int m_s \boldsymbol{v}\boldsymbol{v} f_s(\boldsymbol{r}, \boldsymbol{v}, t) d^3\boldsymbol{v} \ . \tag{5}$$

The physical meaning of the higher moments becomes more clear if we work in the rest frame of the fluid. The relative velocity of particles in this frame is

$$\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{V}_s(\boldsymbol{r}, t) \ . \tag{6}$$

Now, the stress tensor can be rewritten as

$$\mathbf{T}_s = \mathbf{P}_s + m_s n_s \boldsymbol{V}_s \boldsymbol{V}_s \ , \tag{7}$$

where the pressure tensor is

$$\mathbf{P}_s(\boldsymbol{r}, t) = \int m_s \boldsymbol{w}\boldsymbol{w} f_s(\boldsymbol{r}, \boldsymbol{v}, t) d^3\boldsymbol{v} \ . \tag{8}$$

The scalar pressure P_s is given by averaging the diagonal component of the pressure tensor, or

$$P_s(\boldsymbol{r},t) = \frac{1}{3}\mathrm{Tr}(\mathbf{P}_s) = \frac{1}{3}\int m_s w_s^2 f(\boldsymbol{r},\boldsymbol{v},t) d^3\boldsymbol{v} \ , \tag{9}$$

and the tensor pressure can be rewritten as

$$\mathbf{P}_s = P_s \mathsf{I} + \mathbf{\Pi} \ , \tag{10}$$

where $\mathbf{\Pi}$ is called the viscous stress tensor. We can also define internal energy density as

$$\epsilon_s(\boldsymbol{r},t) = \frac{1}{2}\mathrm{Tr}(\mathbf{P}_s) = \frac{3}{2}P_s \ . \tag{11}$$

We note that if the distribution function is isotropic and Maxwellian, it is then uniquely specified by n_s $\boldsymbol{V}_s$, and P_s (or temperature T_s). In this case, the pressure tensor is diagonal, with

$$\mathbf{P}_s(\boldsymbol{r},t) = P_s \mathsf{I} \ , \quad P_s = n_s T_s \ , \quad \epsilon_s = \frac{3}{2} n_s T_s \ . \tag{12}$$

where I is the identity tensor. This is the basis of fluid description of plasmas.

There is also an important third moment measuring the *energy flux density*

$$\boldsymbol{K}_s(\boldsymbol{r},t) = \frac{1}{2}\int m_s v^2 \boldsymbol{v} f_s(\boldsymbol{r},\boldsymbol{v},t) d^3\boldsymbol{v} \ . \tag{13}$$

Similarly, it can be rewritten as

$$\boldsymbol{K}_s = \boldsymbol{q}_s + \mathbf{P}_s \cdot \boldsymbol{V}_s + \epsilon_s \boldsymbol{V}_s + \frac{1}{2} m_s n_s V_s^2 \boldsymbol{V}_s = \boldsymbol{q}_s + \left[\left(\epsilon_s + \frac{1}{2} m_s n_s V_s^2\right)\mathsf{I} + \mathbf{P}_s\right] \cdot \boldsymbol{V}_s \ , \tag{14}$$

where

$$\boldsymbol{q}_s(\boldsymbol{r},t) = \frac{1}{2}\int m_s w^2 \boldsymbol{w} f_s(\boldsymbol{r},\boldsymbol{v},t) d^3\boldsymbol{v} \ . \tag{15}$$

Note that $\boldsymbol{q}_s$ vanishes for a Maxwellian distribution, and other terms simply correspond to fluxes of internal energy and bulk motion, as well as the PdV work.

Moments of the Collision Operator

From the Fokker-Planck equation, we see that the collision term takes the form of

$$\mathcal{C}(f_s) = \frac{\partial}{\partial \boldsymbol{v}} \cdot \boldsymbol{J}_s \ . \tag{16}$$

Clearly, the zeroth moment of the collision operator is zero, which describes particle number conservation.

The first order moment of the collision operator gives frictional force between particle species

$$\int d^3\boldsymbol{v} m_s \boldsymbol{v} \mathcal{C}(f_s) = \sum_{s'} \int d^3\boldsymbol{v} m_s \boldsymbol{v} C_{ss'}(f_s, f_{s'}) \equiv \sum_{s'} \boldsymbol{R}_{ss'} \ . \tag{17}$$

The symmetry of the collision operator (e.g., in Landau form) leads to $\boldsymbol{R}_{ss'} = -\boldsymbol{R}_{s's}$, which describes momentum conservation. For brevity, we can define $\boldsymbol{R}_s \equiv \sum_{s'} \boldsymbol{R}_{ss'}$, and momentum conservation guarantees $\sum_s \boldsymbol{R}_s = 0$.

The second moment of the collision operator with $m_s v^2/2$ describes the energy transfer between particle species. It is again convenient to decompose $\boldsymbol{v} = \boldsymbol{V}_s + \boldsymbol{w}$, and the result is

$$\frac{1}{2}\int d^3\boldsymbol{v} m_s (V_s^2 + w^2 + 2\boldsymbol{V}_s \cdot w)\mathcal{C}(f_s) = \sum_{s'} (\boldsymbol{V}_s \cdot \boldsymbol{R}_{ss'} + Q_{ss'}) \ , \tag{18}$$

where

$$Q_{ss'} \equiv \frac{1}{2}\int d^3\boldsymbol{v} m_s w^2 C_{ss'} \ , \tag{19}$$

which describes the rate of kinetic energy change experienced by species s in its rest frame due to collisions with species s'. For brevity, we can define $Q_s \equiv \sum_{s'} Q_{ss'}$. Energy conservation can be expressed as

$$\sum_s (\boldsymbol{V}_s \cdot \boldsymbol{R}_s + Q_s) = 0 \ , \tag{20}$$

and in the case of simple electron-proton plasmas, it reads

$$Q_i + Q_e = (\boldsymbol{V}_i - \boldsymbol{V}_e) \cdot \boldsymbol{R}_e = \frac{\boldsymbol{J}}{ne} \cdot \boldsymbol{R}_e \ . \tag{21}$$

This corresponds to Joule heating.

Moments of the Kinetic Equation

To take moments of the kinetic equation, it is convenient to write it in conservative form

$$\frac{\partial f_s}{\partial t} + \nabla \cdot (\boldsymbol{v} f_s) + \nabla_{\boldsymbol{v}} \cdot (\boldsymbol{a}_s f_s) = \mathcal{C}[f_s] \ , \tag{22}$$

where

$$\boldsymbol{a}_s \equiv \frac{q_s}{m_s}\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}\right) \ . \tag{23}$$

Taking the zeroth moment, we have the continuity equation

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \boldsymbol{V}_s) = 0 \ . \tag{24}$$

This holds separately for electrons and ions. Because the ions carry almost all the inertia, the fluid density and velocity is essentially $\rho \approx n_i m_i$, $\boldsymbol{V} \approx \boldsymbol{V}_i$. We then obtain

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{V}) = 0 \ . \tag{25}$$

This is the MHD continuity equation.

Taking the first velocity moment, we have

$$m_s \frac{\partial n_s \boldsymbol{V}_s}{\partial t} + \nabla \cdot \mathbf{T}_s = q_s n_s \left(\boldsymbol{E} + \frac{\boldsymbol{V}_s}{c} \times \boldsymbol{B}\right) + \boldsymbol{R}_s \ , \tag{26}$$

or

$$m_s n_s \frac{d\boldsymbol{V}_s}{dt} = -\nabla \cdot \mathbf{P}_s + q_s n_s \left(\boldsymbol{E} + \frac{\boldsymbol{V}_s}{c} \times \boldsymbol{B}\right) + \boldsymbol{R}_s \ , \tag{27}$$

where we have used convective derivative

$$d/dt \equiv \partial/\partial t + \boldsymbol{V}_s \cdot \nabla \ . \tag{28}$$

Again, it holds separately for electrons and ions. On scales much larger than the Debye length, quasi-neutrality guarantees $n_i = n_e$. We can sum over the electron and ion momentum equations, and ignore electron inertia. Note that the frictional forces cancel because $\boldsymbol{R}_i = -\boldsymbol{R}_e$, and electric fields cancel, we obtain

$$\rho \frac{d\boldsymbol{V}}{\partial t} = -\nabla \cdot (\mathbf{P}_i + \mathbf{P}_e) + \frac{1}{c}\boldsymbol{J} \times \boldsymbol{B} = -\nabla \cdot \left(\mathbf{P} + \frac{B^2}{8\pi}\mathsf{I} - \frac{\boldsymbol{B}\boldsymbol{B}}{4\pi}\right) , \tag{29}$$

where $\mathbf{P} \equiv \mathbf{P}_i + \mathbf{P}_e$ is the total pressure tensor. This is the momentum equation for MHD.

Separately, the electron momentum equation gives the generalized Ohm's law, which feeds into the MHD induction equation

$$m_e n_e \frac{d\boldsymbol{V}_e}{dt} + \nabla \cdot \mathbf{P}_e = -en_e\left(\boldsymbol{E} + \frac{\boldsymbol{V}_e}{c} \times \boldsymbol{B}\right) + \boldsymbol{R}_e , \tag{30}$$

or

$$\boldsymbol{E} = -\frac{\boldsymbol{V}_i}{c} \times \boldsymbol{B} + \frac{\boldsymbol{R}_e}{en_e} + \frac{1}{en_e}\frac{\boldsymbol{J} \times \boldsymbol{B}}{c} - \frac{1}{en_e}\nabla \cdot \mathbf{P}_e - \frac{m_e}{e}\frac{d\boldsymbol{V}_e}{dt} , \tag{31}$$

where the frictional force $\boldsymbol{R}_e \propto \boldsymbol{J}/\sigma$ is the source of resistivity. If we are interested in scales much larger than the electron inertial length, we can ignore the electron inertia term (essentially, electrons respond very rapidly to shortcut any additional electric field), and

$$\boldsymbol{E} = -\frac{\boldsymbol{V}}{c} \times \boldsymbol{B} + \boldsymbol{J}/\sigma + \frac{1}{en_e}\frac{\boldsymbol{J} \times \boldsymbol{B}}{c} - \frac{1}{en_e}\nabla \cdot \mathbf{P}_e . \tag{32}$$

This leads to the induction equation in MHD, generalized to include the Hall and electron pressure gradient terms (they again can be dropped if we are interested in scales much larger than the ion inertial length).

Taking the second moment with $m_s v^2/2$, we have

$$\frac{\partial}{\partial t}\left(\frac{3}{2}P_s + \frac{1}{2}m_s n_s V_s^2\right) + \nabla \cdot \boldsymbol{K}_s - q_s n_s \boldsymbol{E} \cdot \boldsymbol{V}_s = \boldsymbol{V}_s \cdot \boldsymbol{R}_s + Q_s , \tag{33}$$

or

$$\frac{d}{dt}\left(\frac{3}{2}P_s + \frac{1}{2}m_s n_s V_s^2\right) + \nabla \cdot (\mathbf{P}_s \cdot \boldsymbol{V}_s) + \left(\frac{3}{2}P_s + \frac{1}{2}m_s n_s V_s^2\right)\nabla \cdot \boldsymbol{V}_s + \nabla \cdot \boldsymbol{q}_s = q_s n_s \boldsymbol{E} \cdot \boldsymbol{V}_s + \boldsymbol{V}_s \cdot \boldsymbol{R}_s + Q_s , \tag{34}$$

To simplify it further, we can dot $\boldsymbol{V}_s$ on both sides of the momentum equation (27), subtract it from the above equation, and make use of the continuity equation (24), to obtain

$$\frac{3}{2}\frac{dP_s}{dt} + \frac{3}{2}P_s \nabla \cdot \boldsymbol{V}_s + \mathbf{P}_s : \nabla \boldsymbol{V}_s + \nabla \cdot \boldsymbol{q}_s = Q_s . \tag{35}$$

Summing again over contributions from both electrons and ions, and using (14) for $\boldsymbol{K}_s$, we have

$$\frac{3}{2}\frac{dP}{dt} + \frac{3}{2}P \nabla \cdot \boldsymbol{V} + \mathbf{P} : \nabla \boldsymbol{V} + \nabla \cdot \boldsymbol{q} = \frac{\boldsymbol{J}}{en} \cdot \boldsymbol{R}_e . \tag{36}$$

where $\boldsymbol{q} = \boldsymbol{q}_i + \boldsymbol{q}_e$ is the total heat flux. If we assume the plasma velocity distribution is isotropic, as in MHD, we have $\mathbf{P} = P\mathsf{I}$, and $\epsilon = (3/2)P$, the above equation reduces to

$$\frac{3}{2}\frac{dP}{dt} + \frac{5}{2}P \nabla \cdot \boldsymbol{V} + \nabla \cdot \boldsymbol{q} = \frac{\boldsymbol{J}}{en} \cdot \boldsymbol{R}_e . \tag{37}$$

Together with the continuity equation, $d\rho/dt + \rho\nabla\cdot\boldsymbol{V} = 0$, and if we further ignore heat conduction and internal heating, we find

$$\frac{d}{dt}\left(\frac{P}{\rho^{5/3}}\right) = 0 \ . \tag{38}$$

This expresses the adiabatic equation of state in ideal MHD.

Fluid Closure

We notice from the derivation of fluid equations above that the system of equations is incomplete. The continuity equation describes the evolution of fluid density n_s, the zeroth moment of the distribution function f_s, but it depends on fluid velocity $\boldsymbol{V}_s$, which is the first moment on f_s. The momentum equation describes how $\boldsymbol{V}_s$ should evolve, but depends on the next order moment, the pressure tensor $\mathbf{P}_s$. We have not fully addressed how the pressure tensor should evolve, but only considered the evolution of its trace ϵ_s, the internal energy, and see that its evolution further depends on the third moment, the heat flux $\boldsymbol{q}_s$. This process can continue forever. In order to complete the set of fluid equations, we must incorporate some additional information to truncate the series by expressing the higher-order moments using the lower-order moments. This process is called *closure*.

In ideal MHD, we close the system by assuming that (1) collisions are sufficiently frequent that particle distribution is Maxwellian, and (2) we are interested in low-frequency ($\ll$ collision and gyro frequencies), long-wavelength ($\gg$ Larmor radius and ion inertial length) regimes. Under these assumptions, pressure is treated as a scalar, closed by an adiabatic equation of state (38). Although in practice, (1) is rarely satisfied, ideal MHD can still provide a moderately accurate description of plasmas so long as the distribution function has second moments that can be roughly associated with a temperature, the electrical conductivity is very large, and thermal conductivity is very small.

Weakly Collisional Plasmas

From now on, let us consider the limit where particle gyro-frequency Ω_L is much higher than collision frequency

$$\Omega_L \gg \nu_c \ , \ (\text{equivalently}, \ r_L \ll \lambda_{\rm mfp}) \tag{39}$$

Quite often, this condition is satisfied in astrophysical plasmas. In this case, it can be shown that the distribution function f_s is *gyrotropic* in the leading order (see problem set), which means f is largely independent of the gyrophase ϕ. This allows us to further simplify the equations by averaging over the gyro-phase. In particular, we have

$$\boldsymbol{w}\boldsymbol{w} = \frac{w_\perp^2}{2}(\mathsf{I} - \boldsymbol{b}\boldsymbol{b}) + w_\parallel^2\boldsymbol{b}\boldsymbol{b} \ , \tag{40}$$

where $\boldsymbol{b}$ is the unit vector along the magnetic field. We can then define parallel and perpendicular pressure

$$P_{s,\parallel} \equiv \int m w_\parallel^2 f_s(\boldsymbol{v}) d^3\boldsymbol{v} \ , \quad P_{s,\perp} \equiv \int m\frac{w_\perp^2}{2} f_s(\boldsymbol{v}) d^3\boldsymbol{v} \ . \tag{41}$$

If we take $\boldsymbol{b}$ to be along the $\hat{z}$-axis, then the pressure tensor in this coordinate system is diagonal, given by

$$\mathbf{P}_s = P_{s,\perp}(\mathsf{I} - \boldsymbol{bb}) + P_{s,\parallel}\boldsymbol{bb} = \begin{pmatrix} P_{s,\perp} & 0 & 0 \\ 0 & P_{s,\perp} & 0 \\ 0 & 0 & P_{s,\parallel} \end{pmatrix} = P_s\mathsf{I} + \frac{1}{3}(P_{s,\perp} - P_{s,\parallel})(\mathsf{I} - 3\boldsymbol{bb}) \ , \tag{42}$$

where total pressure $P_s = (2/3)P_{s,\perp} + (1/3)P_{s,\parallel}$. The extra term in the last equality contributes to the viscous stress tensor $\mathbf{\Pi}_s$. With these expressions, the momentum equation can be rewritten as

$$\rho\frac{d\boldsymbol{V}}{\partial t} = -\nabla\left(P_\perp + \frac{B^2}{8\pi}\right) + \nabla\cdot\left[\boldsymbol{bb}\left(P_\perp - P_\parallel + \frac{B^2}{4\pi}\right)\right] \ . \tag{43}$$

Thus, the dynamics of the plasma is governed by the combined (perpendicular) thermal and magnetic pressure, and a parallel stress consisting of the pressure anisotropy ($P_\perp - P_\parallel$) and magnetic tension. It can also be written as

$$\rho\frac{d\boldsymbol{V}}{\partial t} = -\nabla P + \frac{\boldsymbol{J}\times\boldsymbol{B}}{c} - \nabla\cdot\left[\frac{1}{3}(P_{s,\perp} - P_{s,\parallel})(\mathsf{I} - 3\boldsymbol{bb})\right] \ , \tag{44}$$

where we separate the anisotropic pressure as a viscous stress.

In weakly collisional plasmas, there is no reason that parallel and perpendicular pressure should be equal. In fact, pressure anisotropy can arise simply from fluid motion. This is best illustrated from the double-adiabatic theory, outlined below.

The double-adiabatic theory

Recall from particle orbit theory, magnetic moment $\mu \equiv mv_\perp^2/2B$ is an adiabatic invariant. On scales much larger than particle gyro-radius, conservation of μ implies that

$$p_\perp \propto \rho\langle v_\perp^2\rangle \propto \langle\mu\rangle\rho B \propto \rho B \ . \tag{45}$$

This means that $p_\perp/nB$ is a constant.

Moreover, recall the definition of the longitudinal invariant $J = mv_\parallel l$, where l is the length between two fluid elements. As the fluid travels, l varies accordingly $dl/dt = l(dV/dl)$. It can be shown by combining the continuity and induction equations that $\boldsymbol{B}/n$ satisfies the same equation (along the field line). This can be understood from the fact that total number of particles nlA =const, where A is the cross sectional area enclosed by the line of force. Because of flux freezing, we have $A \sim 1/B$. Hence,

$$p_\parallel \propto \rho\langle v_\parallel^2\rangle \propto \rho\langle J^2/l^2\rangle \propto \rho^3/B^2 \ . \tag{46}$$

Therefore, $p_\parallel B^2/n^3$ is conserved.

The discussion above leads to the "double-adiabatic" equations

$$\frac{d}{dt}\left(\frac{P_\perp}{nB}\right) = 0 \ , \quad \frac{d}{dt}\left(\frac{P_\parallel B^2}{n^3}\right) = 0 \ , \tag{47}$$

which was first derived by Chew, Goldberger & Low (1956). It is the counterpart of the MHD adiabatic equation of state (38) in the collisionless limit. It nicely demonstrates how parallel and perpendicular pressure evolve differently in weakly collisional plasmas. In reality, rate of change in pressure must also include contributions from the heat flux and collisional relaxation.

Kinetic MHD

We have seen that under the assumption (39), particle distribution function is largely gyro-tropic, with distribution function as $f = f(t, \boldsymbol{r}, v_\parallel, v_\perp)$. This allows us to obtain a fairly accurate description of plasmas by averaging the Boltzmann equation over particle gyro-phase. This is reminiscent of the guiding center approximation introduced earlier in the particle orbit theory. In doing so, we are assuming the low-frequency, long-wavelength limit

$$kr_L \ll 1 \ , \quad \omega \ll \Omega_L \ . \tag{48}$$

The outcome of this gyro-phase averaging is the so-called "drift-kinetic" equation, a one-dimensional equation on the distribution function along the field lines. Because the geometry of the field lines is reflected in this equation, the guiding center drift motion is automatically incorporated. The derivation of this equation (Kulsrud, 1964) involves a fair amount of math, and it can be expressed in different forms. The most transparent form of this equation is expressed by using $f = f(v_\parallel, \mu)$ instead of $f(v_\parallel, v_\perp)$ as variables, where μ is magnetic moment. It reads (Kulsrud, 1983)

$$\frac{Df_s}{Dt} + \left(\frac{e}{m}E_\parallel - \mu\nabla_\parallel B - \boldsymbol{b}\cdot\frac{D\boldsymbol{U}_E}{Dt}\right)\frac{\partial f_s}{\partial v_\parallel} = \langle \mathcal{C}(f_s)\rangle \tag{49}$$

where $D/Dt \equiv \partial/\partial t + (\boldsymbol{U}_E + v_\parallel \boldsymbol{b})\cdot\nabla$, with $\boldsymbol{U}_E$ being the $E\times B$ drift velocity, and angle bracket on the right hand side represents gyro-phase averaging.

The physical interpretation of this equation is clear: particles are advected by the $E\times B$ drift velocity; the equation does not explicitly depend on μ as a consequence of μ conservation; particles are subject to parallel forces, including a parallel electric field, magnetic mirror force, and a force due to the change in $E\times B$ drift velocity as the particle travels.

This equation (49) lies in the core of the system known as *Kinetic MHD*, and is often the starting point of many astrophysical plasma problems. The general procedure for using this equation is as follows. From f_s, one calculates $P_{s,\parallel}$ and $P_{s,\perp}$, from which one can evolve velocity $\boldsymbol{V}_s$ using (27), and then $\boldsymbol{B}$ from the induction equation using the Ohm's law. This feeds back to (49), and the system of equations is complete. Essentially, the kinetic MHD formalism is given by a set of fluid equations closed by a 1D kinetic equation along the magnetic field.

Braginskii Equations (two-fluid)

The Braginskii equations are the lowest order closure in the limit where

$$k\lambda_{\rm mfp} \ll 1 \ , \ \text{and} \ \frac{d}{dt} \ll \frac{1}{\tau} \ . \tag{50}$$

Behind this assumption is that all quantities vary slowly in space (small gradients) and time, so that collisional relaxation drives the particle distribution function close to a Maxwellian. These equations were first presented in a celebrated review monograph by Braginskii (1965).

The Braginskii equations are written for the electron and ion fluids separately, which we list below (in the limit of $\Omega_L \tau \gg 1$ for both electrons and ions). The fluid equations for electrons and ions are

$$
\begin{aligned}
\frac{dn}{dt} + n\nabla\cdot\boldsymbol{V}_e &= 0 \ , \\
m_e n \frac{d\boldsymbol{V}_e}{dt} + \nabla P_e + \nabla\cdot\boldsymbol{\Pi}_e + en(\boldsymbol{E} + \boldsymbol{V}_e \times \boldsymbol{B}) &= \boldsymbol{R} \ , \\
\frac{3}{2}\frac{dP_e}{dt} + \frac{5}{2}P_e \nabla\cdot\boldsymbol{V}_e + \boldsymbol{\Pi}_e : \nabla\boldsymbol{V}_e + \nabla\cdot\boldsymbol{q}_e &= Q_e \ ,
\end{aligned} \tag{51}
$$

and

$$
\begin{aligned}
\frac{dn}{dt} + n\nabla\cdot\boldsymbol{V}_i &= 0 \ , \\
m_i n \frac{d\boldsymbol{V}_i}{dt} + \nabla P_i + \nabla\cdot\boldsymbol{\Pi}_i - en(\boldsymbol{E} + \boldsymbol{V}_i \times \boldsymbol{B}) &= -\boldsymbol{R} \ , \\
\frac{3}{2}\frac{dP_i}{dt} + \frac{5}{2}P_i \nabla\cdot\boldsymbol{V}_i + \boldsymbol{\Pi}_i : \nabla\boldsymbol{V}_i + \nabla\cdot\boldsymbol{q}_i &= Q_i \ .
\end{aligned} \tag{52}
$$

Note that $n = n_i = n_e$ by charge neutrality.

The source terms on the right hand sides are

$$
\begin{aligned}
\boldsymbol{R} &= ne\left(\frac{\boldsymbol{J}_\parallel}{\sigma_\parallel} + \frac{\boldsymbol{J}_\perp}{\sigma_\perp}\right) - 0.71nk\nabla_\parallel T_e - \frac{3n}{2|\Omega_e|\tau_e}\boldsymbol{b}\times k\nabla_\perp T_e \ , \\
Q_i &= \frac{3m_e}{m_i}\frac{nk(T_e - T_i)}{\tau_e} \ , \\
Q_e &= -Q_i + \frac{\boldsymbol{J}\cdot\boldsymbol{R}}{ne} \ ,
\end{aligned} \tag{53}
$$

where parallel and perpendicular conductivities are

$$
\sigma_\perp = \frac{ne^2\tau_e}{m_e} \ , \quad \sigma_\parallel = 1.96\sigma_\perp \ . \tag{54}
$$

The symbols $\nabla_\parallel \equiv \boldsymbol{bb}\cdot\nabla$, $\nabla_\perp = \nabla - \nabla_\parallel$ denote gradients parallel and perpendicular to the magnetic field. The first term in $\boldsymbol{F}$ corresponds to parallel and perpendicular electric conductivity. The second term in $\boldsymbol{F}$ is called the *thermal force*, which has its origin in the fact that faster electrons experience less frictional forces (and hence there is a force imbalance across a temperature gradient). The third term is related to the gyro-magnetic particle effect: at a given location across a temperature gradient, particles gyrating from the hotter side on average experience less drag than particles gyrating from the cooler side. The $Q_{i,e}$ terms describe energy exchange between electrons and ions, together with Joule heating which mainly acting on the electrons.

The heat fluxes are given by

$$
\begin{aligned}
\boldsymbol{q}_e &= -\kappa^e_\parallel \nabla_\parallel T_e - \kappa^e_\perp \nabla_\perp T_e - \kappa^e_\times \boldsymbol{b}\times\nabla_\perp T_e - 0.71\frac{kT_e}{e}\boldsymbol{j}_\parallel - \frac{3}{2|\Omega_e|\tau_e e}\boldsymbol{b}\times\boldsymbol{j}_\perp \ , \\
\boldsymbol{q}_i &= -\kappa^i_\parallel \nabla_\parallel T_i - \kappa^i_\perp \nabla_\perp T_i - \kappa^i_\times \boldsymbol{b}\times\nabla_\perp T_i \ ,
\end{aligned} \tag{55}
$$

with thermal conductivities given by

$$
\kappa^e_\parallel = 3.2\frac{n\tau_e kT_e}{m_e} \ , \quad \kappa^i_\parallel = 3.9\frac{n\tau_i kT_i}{m_i} \ , \tag{56}
$$

$$\kappa_\perp^e = 4.7\frac{nkT_e}{m_e\Omega_e^2\tau_e} \ , \quad \kappa_\perp^i = 2\frac{nkT_i}{m_i\Omega_i^2\tau_i} \ , \tag{57}$$

$$\kappa_\times^e = \frac{5nkT_e}{2m_e|\Omega_e|} \ , \quad \kappa_\times^i = \frac{nkT_i}{2m_i|\Omega_i|} \ . \tag{58}$$

Most of the terms have been discussed in the previous lecture, including parallel and perpendicular heat fluxes and the cross thermal conduction. The last two terms in $\boldsymbol{q}_e$ correspond to transport associated with the relative drift between the electron and ion fluids. While it is straightforward to interpret the fourth term as associated with electron drift parallel to the magnetic field, the fifth term is more tricky. In the presence of perpendicular drift, the drag force experienced by the electrons does not cancel over one gyration, leading a drift in the $\boldsymbol{j}_\perp \times \boldsymbol{B}$ direction.

Finally, the viscous stress is given by

$$\mathbf{\Pi} = \sum_{n=0,4} \mathbf{\Pi}_n \ , \tag{59}$$

where

$$\begin{aligned}
\mathbf{\Pi}_0 &= -\frac{1}{3}\eta_0\Big(\mathsf{I} - 3\boldsymbol{bb}\Big)\Big(\mathsf{I} - 3\boldsymbol{bb}\Big) : \nabla\boldsymbol{V} \ , \\
\mathbf{\Pi}_1 &= -\eta_1\left[\mathsf{I}_\perp \cdot \mathbf{W} \cdot \mathsf{I}_\perp + \frac{1}{2}\mathsf{I}_\perp(\boldsymbol{b} \cdot \mathbf{W} \cdot \boldsymbol{b})\right] \ , \\
\mathbf{\Pi}_2 &= -4\eta_1(\mathsf{I}_\perp \cdot \mathbf{W} \cdot \boldsymbol{bb} + \boldsymbol{bb} \cdot \boldsymbol{W} \cdot \mathsf{I}_\perp) \ , \\
\mathbf{\Pi}_3 &= \frac{\eta_3}{2}(\boldsymbol{b} \times \mathbf{W} \cdot \mathsf{I}_\perp - \mathsf{I}_\perp \cdot \mathbf{W} \times \boldsymbol{b}) \ , \\
\mathbf{\Pi}_4 &= 2\eta_3(\boldsymbol{b} \times \mathbf{W} \cdot \boldsymbol{bb} - \boldsymbol{bb} \cdot \mathbf{W} \times \boldsymbol{b}) \ ,
\end{aligned} \tag{60}$$

and

$$\mathsf{I}_\perp = \mathbf{I} - \boldsymbol{bb} \ , \quad W_{\alpha\beta} \equiv \frac{\partial V_\alpha}{\partial x_\beta} + \frac{\partial V_\beta}{\partial x_\alpha} - \frac{2}{3}\delta_{\alpha\beta}\nabla \cdot \boldsymbol{V} \ . \tag{61}$$

Of course, there are separate stress tensors for the electrons and ions. The corresponding viscosity coefficients are given by

$$\eta_0^e = 0.73n\tau_e kT_e \ , \quad \eta_0^i = 0.96n\tau_i kT_i \ . \tag{62}$$

$$\eta_1^e = 0.51\frac{nkT_e}{\Omega_e^2\tau_e} \ , \quad \eta_1^i = \frac{3nkT_i}{10\Omega_i^2\tau_i} \ . \tag{63}$$

$$\eta_3^e = -\frac{nkT_e}{2|\Omega_e|} \ , \quad \eta_3^i = \frac{nkT_i}{2\Omega_i} \ . \tag{64}$$

In the above, the tensor $\mathbf{\Pi}_0$ is known as *parallel viscosity*, $\mathbf{\Pi}_1$ and $\mathbf{\Pi}_2$ are called *perpendicular viscosity*, which are strongly suppressed by magnetic fields, $\mathbf{\Pi}_3$ and $\mathbf{\Pi}_4$ describe the *gyroviscosity*, which are dissipation-free. In the limit we are considering ($\Omega_L\tau \gg 1$), parallel viscosity dominates over other terms.

The transport coefficients involve the definition of collision times τ_i and τ_e, given by

$$\begin{aligned}
\tau_e &= \frac{3}{4\sqrt{2\pi}\ln\Lambda}\frac{(kT_e)^{3/2}m_e^{1/2}}{ne^4} \ , \\
\tau_i &= \frac{3}{4\sqrt{\pi}\ln\Lambda}\frac{(kT_i)^{3/2}m_i^{1/2}}{ne^4} \ .
\end{aligned} \tag{65}$$

They are again derived from the Fokker-Planck calculations of the electron-ion collision and ion-ion collision times.

Braginskii Equations (single-fluid)

In most situations, as we are interested in scales much larger than the ion inertia length, the velocity difference in $\boldsymbol{V}_e$ and $\boldsymbol{V}_i$ from the current is small (i.e., the Hall term, together with the pressure gradient term in the generalized Ohm's law can be neglected), and we can sum over the electron and ion equations to obtain a system of single-fluid equations. In this set of equations, we can ignore terms associated with electron inertia. Heat conduction and resistivity are dominated by parallel transport by the electrons, while viscosity is mainly contributed by the parallel viscosity from the ions, and we can ignore other sub-dominant terms. The result is

$$\begin{aligned}
\frac{d\rho}{\partial t} &= -\rho\nabla\cdot\boldsymbol{V} \ , \\
\rho\frac{d\boldsymbol{V}}{\partial t} &= -\nabla P + \nabla\cdot\left[\frac{1}{3}\eta_0^i\Big(\mathsf{I}-3\boldsymbol{bb}\Big)\Big(\mathsf{I}-3\boldsymbol{bb}\Big):\nabla\boldsymbol{V}\right] + \frac{1}{c}\boldsymbol{J}\times\boldsymbol{B} \ , \\
\frac{3}{2}\frac{dP}{dt} &= -\frac{5}{2}P\nabla\cdot\boldsymbol{V} + \frac{1}{3}\eta_0^i\left[\Big(\mathsf{I}-3\boldsymbol{bb}\Big):\nabla\boldsymbol{V}\right]^2 + \nabla\cdot(\kappa_\parallel^e\boldsymbol{bb}\cdot\nabla T_e) + \boldsymbol{E}\cdot\boldsymbol{J} \ ,
\end{aligned} \tag{66}$$

where the electric and magnetic fields are determined by

$$\boldsymbol{E} = -\frac{\boldsymbol{V}_i}{c}\times\boldsymbol{B} + \left(\frac{\boldsymbol{J}_\parallel}{\sigma_\parallel}+\frac{\boldsymbol{J}_\perp}{\sigma_\perp}\right) \ , \quad \frac{\partial\boldsymbol{B}}{\partial t} = -c\nabla\times\boldsymbol{E} \ . \tag{67}$$

where coefficients $\sigma_\parallel, \sigma_\perp$, $\kappa_\parallel^e$ and η_0^i have been given previously in (54), (56), (62).

By comparing the momentum equation above and (44), we find

$$P_\perp - P_\parallel = -\eta_0^i(\mathsf{I}-3\boldsymbol{bb}):\nabla\boldsymbol{V} \approx \frac{P_i}{\nu_i}(\mathsf{I}-3\boldsymbol{bb}):\nabla\boldsymbol{V} \ , \tag{68}$$

where we have used (62), and $P_i = nkT_i$, $\nu_i = 1/\tau_i$. This describes the relaxation process: pressure anisotropy is developed by velocity gradient (with respect to the magnetic fields), while collision drives the system towards pressure isotropy. Higher collision frequency leads to lower level of pressure anisotropy.

Anisotropic thermal conduction and anisotropic viscosity are the two most salient features of the Braginskii MHD. In weakly collisional systems, thermal conduction along the magnetic field can be very efficient due to the long mean free path, and magnetic field lines become isotherms; temperature gradient is almost always perpendicular to the field lines, due to inefficient perpendicular thermal conduction. Similarly, with large parallel viscosity and negligible perpendicular viscosity, field lines become "isotachs": flow speed tends to be constant along a field line.