Astronomy 253 (Spring 2016)
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Mar. 28, 2016

Instabilities in Weakly Collisional Plasmas

Fluid Derivation of The Firehose Instability

The starting point is the momentum equation with anisotropic pressure

$$\rho\frac{d\boldsymbol{v}}{dt}=-\nabla\left(P_\perp+\frac{B^2}{8\pi}\right)+\nabla\cdot\left[\boldsymbol{bb}\left(P_\perp-P_\parallel+\frac{B^2}{4\pi}\right)\right]\ , \tag{1}$$

and the induction equation

$$\frac{\partial\boldsymbol{B}}{\partial t}=\nabla\times(\boldsymbol{v}\times\boldsymbol{B})\ . \tag{2}$$

In the background state, $\boldsymbol{B}=\boldsymbol{B}_0$ is homogeneous, and $\boldsymbol{v}=0$. We consider perturbations denoted with symbol δ, and decompose all quantities in the form of $\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)]$. Furthermore, all vectors are decomposed into components parallel and perpendicular to the background field.

The linearized induction equation becomes

$$-\omega\delta\boldsymbol{B}=\boldsymbol{k}\times(\delta\boldsymbol{v}\times\boldsymbol{B})=(\boldsymbol{k}\cdot\boldsymbol{B})\delta\boldsymbol{v}-\boldsymbol{B}(\boldsymbol{k}\cdot\delta\boldsymbol{v})=k_\parallel B\delta\boldsymbol{v}_\perp-\boldsymbol{B}(\boldsymbol{k}_\perp\cdot\delta\boldsymbol{v}_\perp)\ . \tag{3}$$

Geometrically, we have

$$\delta\boldsymbol{B}=\delta B_\parallel\boldsymbol{b}+\delta\boldsymbol{B}_\perp\ ,\quad \delta\boldsymbol{b}=\frac{\delta\boldsymbol{B}_\perp}{B}\ . \tag{4}$$

Thus from the induction equation, we obtain

$$\delta\boldsymbol{b}=-\frac{k_\parallel}{\omega}\delta\boldsymbol{v}_\perp\ ,\quad \frac{\delta B_\parallel}{B}=\frac{\boldsymbol{k}_\perp\cdot\delta\boldsymbol{v}_\perp}{\omega}\ . \tag{5}$$

From divergence free requirement, we have (which is in fact implied from the induction equation)

$$\boldsymbol{k}\cdot\delta\boldsymbol{B}=0=(\boldsymbol{k}\cdot\boldsymbol{b})\delta B_\parallel+(\boldsymbol{k}\cdot\delta\boldsymbol{b})B\ \Rightarrow\ \boldsymbol{k}\cdot\delta\boldsymbol{b}=-\frac{k_\parallel\delta B_\parallel}{B}\ . \tag{6}$$

For the momentum equation, we note that we also need to take into account changes in field direction, reflected in the $\boldsymbol{b}$ vector. The result is

$$\begin{aligned}-\omega\rho\delta\boldsymbol{v}&=-\boldsymbol{k}\left(\delta P_\perp+\frac{B\delta B_\parallel}{4\pi}\right)+\boldsymbol{k}\cdot\left[(\delta\boldsymbol{b}\boldsymbol{b}+\boldsymbol{b}\delta\boldsymbol{b})\left(P_\perp-P_\parallel+\frac{B^2}{4\pi}\right)+\boldsymbol{bb}\left(\delta P_\perp-\delta P_\parallel+\frac{B\delta B_\parallel}{2\pi}\right)\right]\\&=-\boldsymbol{k}_\perp\left(\delta P_\perp+\frac{B\delta B_\parallel}{4\pi}\right)-\boldsymbol{b}k_\parallel\left[\delta P_\parallel+(P_\perp-P_\parallel)\frac{\delta B_\parallel}{B}\right]+\delta\boldsymbol{b}k_\parallel\left(P_\perp-P_\parallel+\frac{B^2}{4\pi}\right)\ .\end{aligned} \tag{7}$$

Although it seems quite complex, if we focus on Alfvénic perturbations, we can reduce it substantially. First, we only need to look at the perpendicular component of this equation, which becomes

$$-\omega\rho\delta\boldsymbol{v}_\perp=-\boldsymbol{k}_\perp\left(\delta P_\perp+\frac{B\delta B_\parallel}{4\pi}\right)+k_\parallel\delta\boldsymbol{b}\left(P_\perp-P_\parallel+\frac{B^2}{4\pi}\right)\ . \tag{8}$$

Next, Alfvénic perturbations are in the direction of $\boldsymbol{k} \times \boldsymbol{b}$. Also from (5) that $\delta \boldsymbol{b} \parallel \delta \boldsymbol{v}_\perp$, we see that the left hand side and the second term on the right are decoupled from the rest of the system. (This can be seen by taking the cross product on $\boldsymbol{k}_\perp$ on both sides.) Leaving only these two terms, we arrive at the dispersion relation

$$\omega^2 \rho = k_\parallel^2 \left(P_\perp - P_\parallel + \frac{B^2}{4\pi} \right) . \tag{9}$$

Clearly, it reduces to the dispersion relation for Alfvén waves when $P_\parallel = P_\perp$. However, when $P_\parallel - P_\perp > B^2/4\pi$, the system becomes unstable. This is the *firehose* instability.