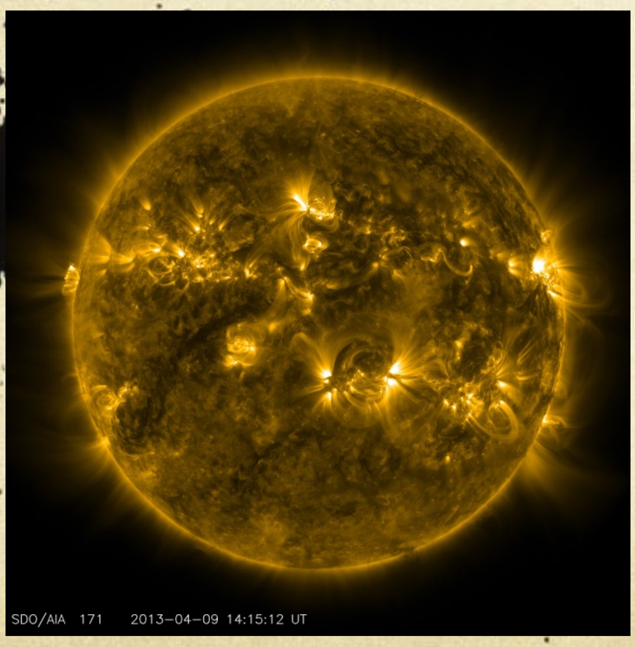
Magneto-rotational Instability in Accretion Disks



The instability can make accretion disks turbulent, providing the source of anomalous viscosity.

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Standard Accretion Disk Dilemma

- It is not as easy as it may seem to be for the disk to accrete!
- → mechanism needed for matter to lose angular momentum
- → viscosity helps the angular momentum transport
- Source of viscosity? Anomalous α assumed and question pended.

Viscosity

- Viscosity characterize how fast the momentum exchanges at typical length scale.
- For thermal motion in molecular gas, viscosity :
$$\nu = v_{thermal} \cdot \lambda_{mean\ free\ path}$$
- If the momentum exchange in accretion disks happens at molecular scale:
- Viscous time scale : $\frac{R^2}{\nu} \sim \frac{R^2}{v_{th}\,\lambda_{mfp}} \sim 10^{11}\ yr$ (cold protoplanetary disk) > age of the universe !
- Need something at larger scale.

Turbulence

- Turbulence involves momentum exchange through large-scale mixing, which mimics viscosity
- Emperically, incompressible pipe flow becomes turbulent when the Reynolds number $Re = \frac{uL}{\nu} > 10^3$
- For a protoplanetary disk (as an example):
  - $Re = \frac{v_\phi H}{\nu} \sim 10^{11} \gg 10^3$
- Seems like the accretion disk might be turbulent. Not sure if apply to compressible and differentially rotating plasma.

Rayleigh Criterion

- Linear analysis (Rayleigh, 1921) shows that differentially rotating hydrodynamic flow is stable when

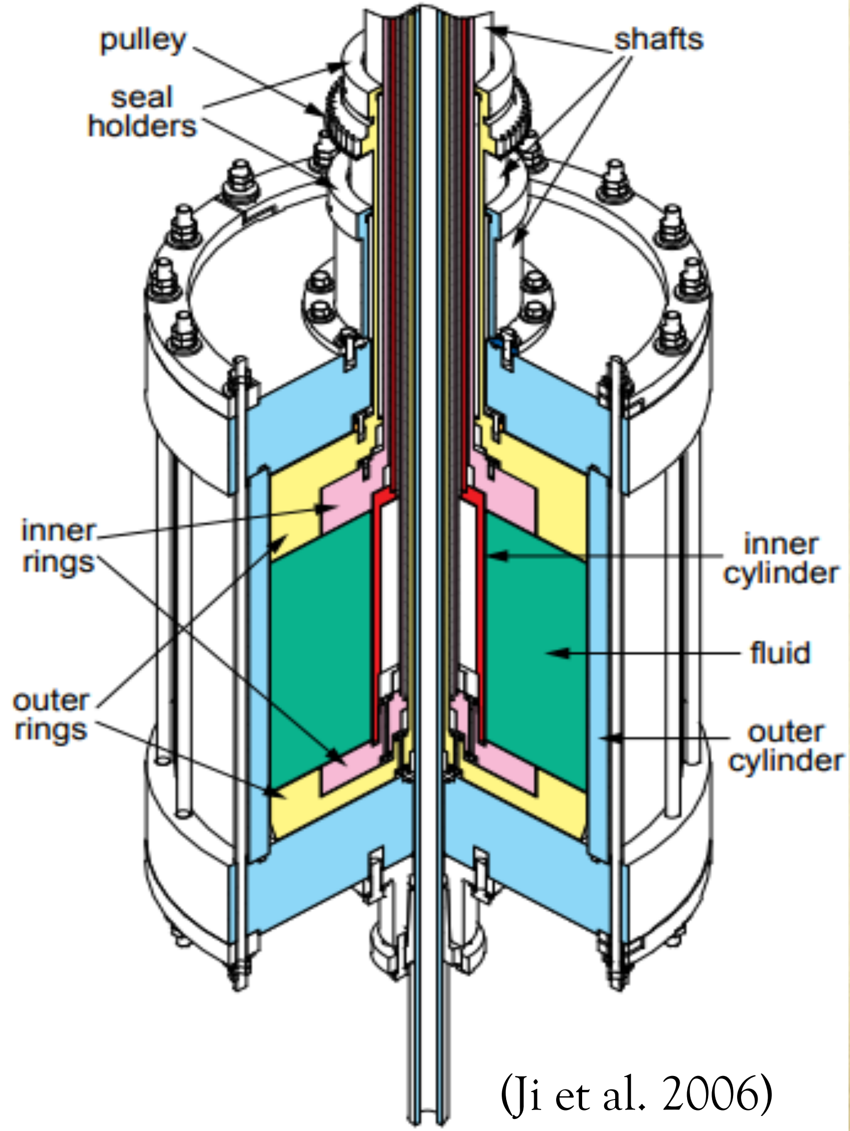
$$\frac{dr^2\Omega}{dr} > 0$$

- Keplarian disks should be linearly stable according to the Rayleigh criterion.

- Which one is true? ($Re = \frac{v_\phi H}{\nu} \gg 10^3$ vs. $\frac{dr^2\Omega}{dr} > 0$)

Experimental evidence

- Ji et al report a laboratory experiment, demonstrating that non-magnetic quasi-Keplerian flows at $Re \sim 10^7$ are essentially steady.
- Indirectly support the Rayleigh criterion and we need something else for turbulence.



(Ji et al. 2006)

Magneto-rotational Instability

- First noticed in non-astrophysical context when considering the stability of Couette flow (S. Chandrasekhar, Evgeny Velikhov).
- Balbus & Hawley (1991) re-discovered the instability and brought it to the context of astrophysical accretion disk
- In the presense of magnetic field, no matter how weak it is, the stablility criterion changes qualitatively from $\frac{dr^2\Omega}{dr} > 0$ to $\frac{d\Omega}{dr} > 0$

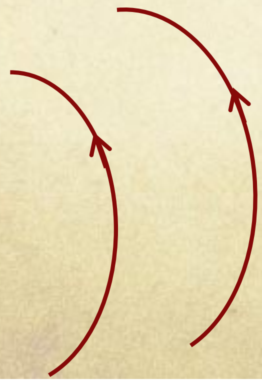
The Magnetic field in astro-plasma

- From magneto-hydrodynamics : the magnetic field line is 'frozen' in the conducting fluid
- The Lorentz force can be decomposed to "Magnetic tension" and "Magnetic pressure"

$$f_L = \frac{1}{c} j_e \times B = \frac{1}{4\pi} (\nabla \times B) \times B$$
$$= \frac{1}{4\pi} (B \cdot \nabla) B - \frac{1}{8\pi} \nabla (|B|^2)$$

Magnetic tension

If the field lines have curvature, it tends to strength the lines.

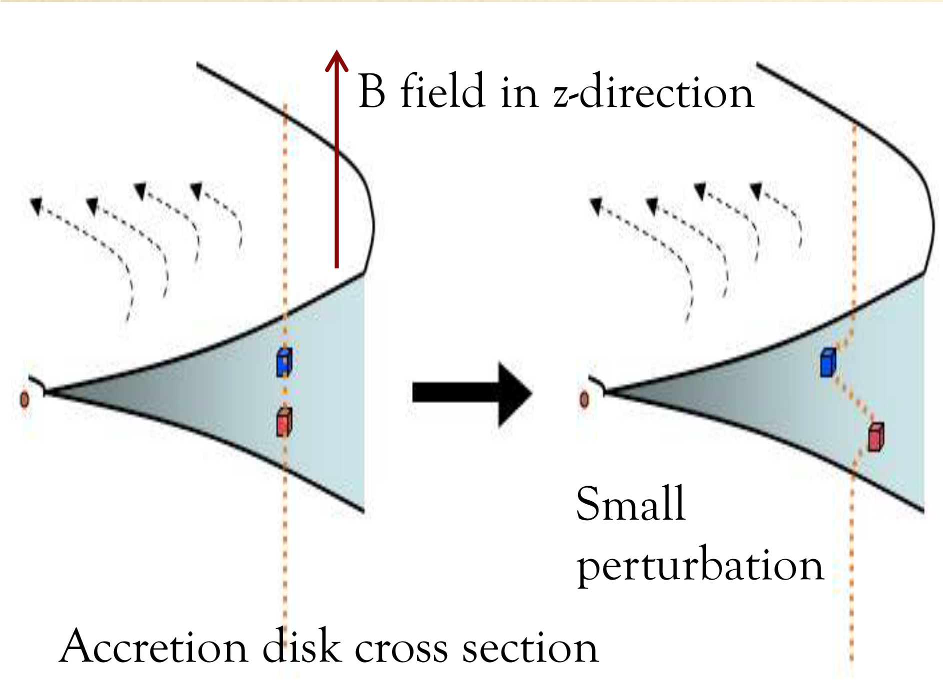


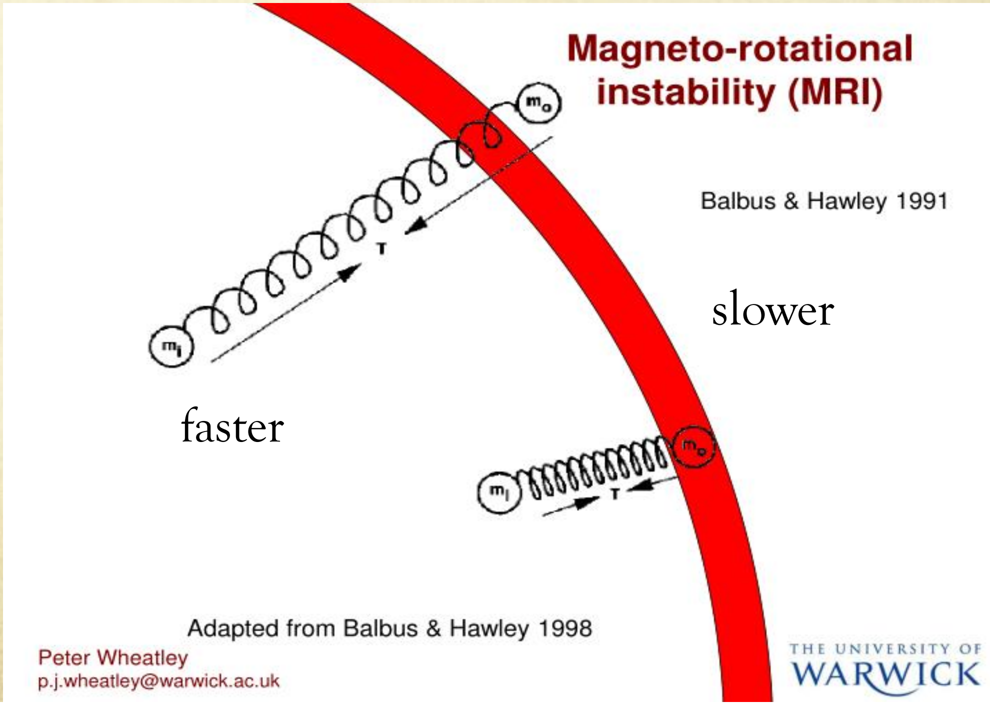
Magnetic pressure

If the field lines has a gradient in strength

MRI physics picture

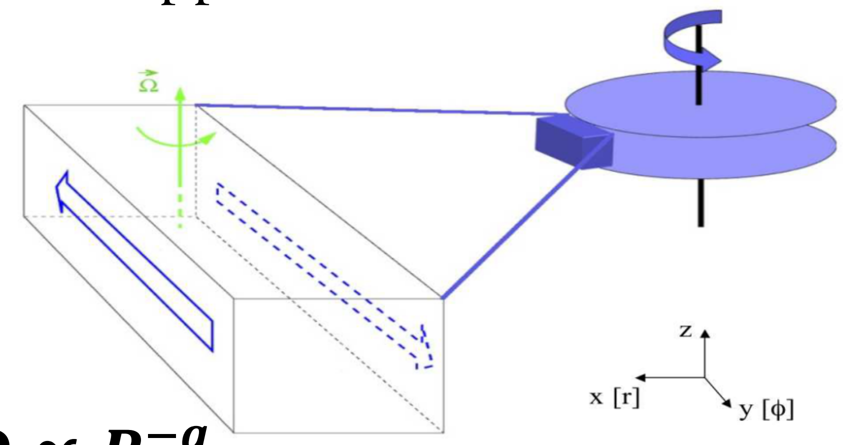
- Given a small perturbation, the field line is bent and tend to drag it back.
  - Perturbation can always be decomposed to this ‘spring-like’ modes.
- Inner ball : moving faster and losing angular momentum
- Outer ball : moving slower and gaining angular momentum





The MRI in accretion disk

Analysis is the most convenient in a “local” frame corotating with a fluid element at a certain radius: shearing-sheet approximation



$\boldsymbol{\Omega \propto R^{-q}}$

- Mass conservation:
  - $\frac{\partial \rho}{\partial t} + \nabla \rho \cdot v = 0$
- Force balance:
  - $\rho\left[\frac{\partial v}{\partial t} + (v \cdot \nabla)v\right] = -\nabla P + \frac{1}{4\pi}(\nabla \times B) \times B + 2q\rho\Omega_0^2 x e_x - 2\rho\Omega_0 \times v$

    Acceleration | Pressure | Lorentz force | Tidal force | Coriolis force
- The magnetic field line frozen with the conducting fluid:
  - $\frac{\partial B}{\partial t} = \nabla \times (v \times B)$

Giving a small perturbation ...

- Solve for a simple case: uniform density, only Bz
- Small perturbation $B = B_0 + B'(t), v = v_0 + v'(t),$
- Solution in Fourier space takes the form :
  - $\boldsymbol{X'} = \overline{\boldsymbol{X'}}\boldsymbol{e}^{\boldsymbol{i(wt-kz)}}$
  - $\frac{\partial}{\partial t} \to iw\,, \quad \nabla \to ik$
  - When w is real, the perturbation propagate as wave
  - When w is imaginary, the perturbation grows with time, namely, instability generated.

Dispersion Relation

- $w^4 - w^2[2k^2 v_A^2 + \kappa^2] + k^2 v_A^2 [k^2 v_A^2 - 2q\Omega^2] = 0$
  - $\kappa^2 = 2(2-q)\Omega^2$ square of the epicyclic frequency
  - $v_A = B_0/\sqrt{\rho_0}$ the Alfven (magnetic wave) velocity
  - $\Omega \propto R^{-q}$ by definition

In the absence of a magnetic field

$$\boldsymbol{w^4 - w^2\left[2k^2v_A^2 + \kappa^2\right] + k^2v_A^2\left[k^2v_A^2 - 2q\Omega^2\right] = 0}$$

- $v_A = 0$ (magnetic wave velocity)
- Dispersion relation reduces to :
- $w^2 - \kappa^2 = 0$
  - Require $w^2 > 0$ to be stable
  - → $\kappa^2 > 0$
  - → $q < 2$
  - → $\frac{dr^2\Omega}{dr} > 0$
- We just derived Rayleigh criterion!

$\kappa^2 = 2(2-q)\Omega^2$
$v_A = B_0/\sqrt{\rho_0}\ \Omega \propto R^{-q}$

In the presence of a magnetic field

$$\boldsymbol{w^4 - w^2\left[2k^2 v_A^2 + \kappa^2\right] + k^2 v_A^2\left[k^2 v_A^2 - 2q\Omega^2\right] = 0}$$

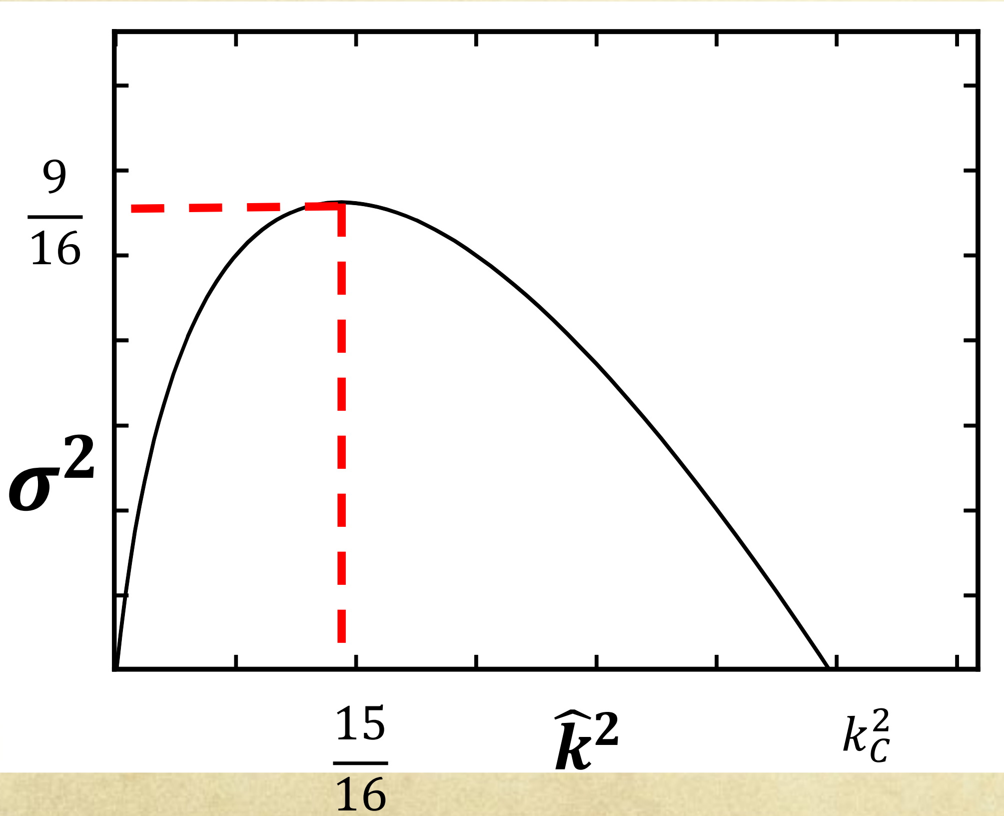
- $v_A^2 > 0$:
- stability condition become $k^2 v_A^2 - 2q\Omega^2 \geq 0$
  - → $q < 0$
  - → $\frac{d\Omega}{dr} > 0$

$\kappa^2 = 2(2-q)\Omega^2$
$v_A = B_0/\sqrt{\rho_0}\ \Omega \propto R^{-q}$

- This condition completely changed the physics picture because now the criterion does not hold for common Keplerian orbiting accretion disk.

MRI properties

- Especially, if $q = \frac{3}{2}$, solve for the $\sigma = \frac{iw}{\Omega_R}$, $\hat{k} = \frac{k v_A}{\Omega_K}$
  - $\sigma^2 = \frac{-(2\hat{k}^2+1) + \sqrt{16\hat{k}^2+1}}{2}$

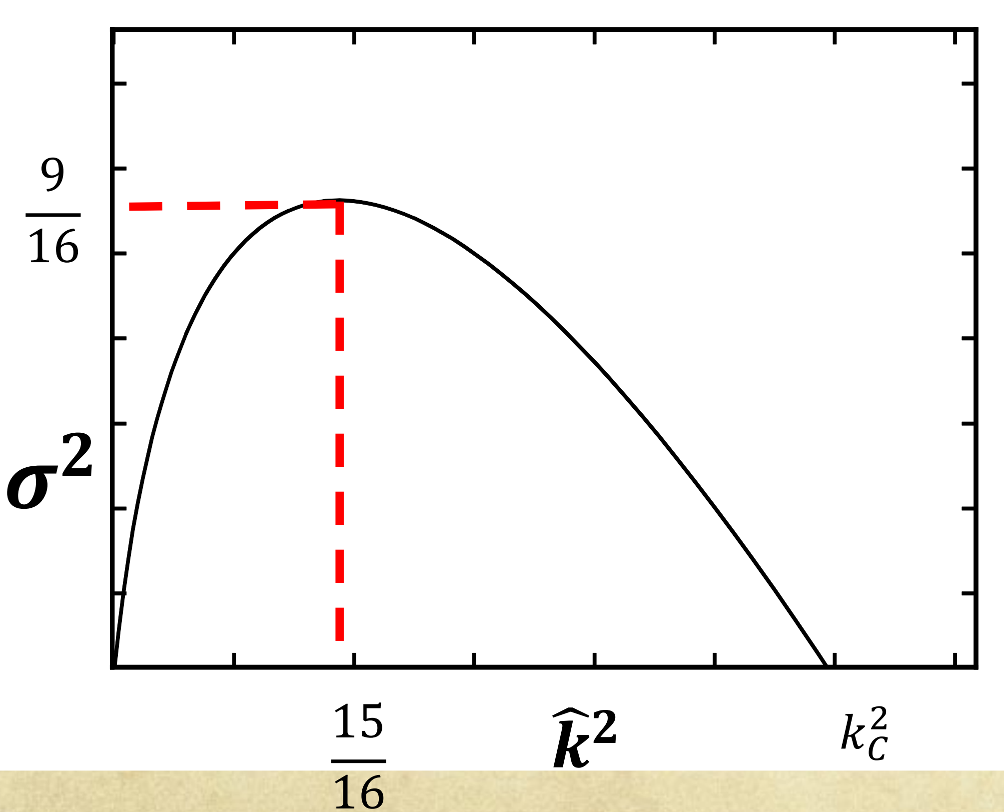


When the $\sigma^2 > 0$, perturbation become unstable

Maximum growth rate

MRI properties

- Especially, if $q = \frac{3}{2}$, solve for the $\sigma = \frac{iw}{\Omega_R}$, $\hat{k} = \frac{kv_A}{\Omega_K}$
  - $\sigma^2 = \frac{-(2\hat{k}^2+1) + \sqrt{16\hat{k}^2+1}}{2}$



It requires the wavelength of the unstable mode to be smaller than the scale height of the disk

$$k_C^2 = \frac{2q\Omega^2}{v_A^2} > \frac{2\pi}{H}$$

→ the magnetic field should be **weaker** than a threshold where magnet and thermal energy reaches equilibrium.

Summary

- Turbulence (instead of molecular viscosity) is needed to explain the accretion phenomenon in astrophysical disks.
- Accretion disks are hydrodynamically stable according to the Rayleigh criterion.
- The presence of magnetic fields qualitatively changes the stability criterion to $\frac{d\Omega}{dr} > 0$, giving rise to the magneto-rotational instability.
- The MRI is a weak field instability.
- The non-linear evolution of the MRI leads to turbulence, which accounts for the anomalous viscosity in alpha disk models.

References

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- Fromang 2013 – MRI-driven angular momentum transport in protoplanetary disks
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- Ji Hantao et al, 2006 – Hydrodynamic turbulence cannot transport angular momentum effectively in astrophysical disks
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